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Original Paper

Decompositions of $1/n$ into sums and differences of reciprocals of the forms $2/a$ and $3/b$

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ABSTRACT

This paper addresses the elementary problem of decomposing the reciprocal of a positive integer into a specific sum or difference of two reciprocals and computing their total number. We show that, for any positive integer n, the number of decompositions of $1/n$ into the form $2/a \pm 3/b$ is determined by the number of positive divisors of $6n^2$, and we derive general results together with explicit formulas. The method is illustrated by explicit decompositions for a chosen positive integer. Finally, we describe a program implemented in the computer algebra system Maple 2025 that computes these decompositions for any positive integer, and we confirm the theoretical results using the earlier example.

KEYWORDS: reciprocal, Egyptian fraction, greedy algorithm, Erdős–Straus conjecture, prime factorization, Simon's favorite factoring trick, Diophantine equation, divisor function, computer algebra system Maple

JEL CLASSIFICATION: C60

INTRODUCTION

This paper continues the author's previous work [10], which deals with the decompositions of the reciprocal of a positive integer n into the sum and difference of two reciprocals. This study introduces a simple method to determine decompositions of $1/n$ into the sums

$$\frac{1}{n} = \frac{2}{a} + \frac{3}{b} \tag{1}$$

and differences

$$\frac{1}{n} = \frac{2}{c} - \frac{3}{d}, \tag{2}$$

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with positive integers a, b, c, d, where $c < d$. Furthermore, we determine how many decompositions of $1/n$ of the forms (1) and (2) exist for an arbitrary positive integer n. Recall that $1/n$, where n is a nonzero integer, is called the *reciprocal* of n (see, e.g., [12]).

Note that an *Egyptian fraction* is defined as a sum of finitely many rational numbers, each of which can be written in the form $1/q$, where q is a positive integer. For example,

$$\frac{7}{19} = \frac{1}{5} + \frac{1}{10} + \frac{1}{19} + \frac{1}{95} + \frac{1}{190}.$$

One famous unproven statement in number theory is the *Erdős–Straus conjecture* from 1948. The conjecture is that, for every integer $n \geq 2$ there exist positive integers x, y, z for which

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

(see, e.g., [1] and [11]). Online calculators for Egyptian fractions can be found, for example, on [4], [5], and [9]. Reciprocals, their sums, and Egyptian fractions remain of interest to mathematicians, and in recent years several interesting papers have been published on these topics, e.g. [2], [3], [6], [7], [8], and [10].

MATERIAL AND METHODS

The analysis in this study relies exclusively on theoretical number-theoretic tools and symbolic algebraic manipulation. The "material" consists of positive integers, their divisors, and the prime factorization of the composite number $6n^2$, which naturally arises from the Diophantine equation governing the decompositions of $1/n$.

Using Simon's Favorite Factoring Trick, we transform the original decomposition problems into algebraic conditions on the divisors of $6n^2$. This approach yields closed formulas for the number of sum decompositions of the form $2/a + 3/b$, and an inequality together with an auxiliary divisor condition $u < 2n$ for divisor u of $6n^2$ for determining the number of admissible difference decompositions $2/c - 3/d$.

All theoretical findings were verified computationally using a Maple 2025 routine specifically developed to generate, enumerate, and check the decompositions for any prescribed positive integer, with detailed verification performed for the illustrative case $n = 10$.

RESULTS AND DISCUSSION

The solution to the given problem and the corresponding comments are further divided into two cases of decomposition – additive case and differential one.

Additive case

We now derive the number of decompositions of the form (1), that is,

$$\frac{1}{n} = \frac{2}{a} + \frac{3}{b},$$

where a, b are positive integers. Multiplying both sides by nab, we obtain

$$ab = n(2b + 3a)\,,$$

which is equivalent to

$$ab - 2nb - 3na = 0\,.$$

Adding $6n^2$ to both sides and then factoring (by applying *Simon's Favorite Factoring Trick*, SFFT), we obtain

$$(a - 2n)(b - 3n) = 6n^2.$$

Let us denote

$$x = a - 2n > 0\,, \qquad y = b - 3n > 0\,.$$

Then

$$xy = 6n^2.$$

Thus, each positive divisor x of $6n^2$ yields a solution

$$a = x + 2n\,, \qquad b = y + 3n = \frac{6n^2}{x} + 3n\,.$$

This produces all solutions (a, b) of equation (1) in positive integers, i.e., all additive decompositions. If the positive integer n has prime factorization

$$n = \prod_{i=1}^{k} {p_i}^{\alpha_i}\,,$$

then

$$6n^2 = 2^{1+2\alpha_2} 3^{1+2\alpha_3} \prod_{p \neq 2,3} p^{2\alpha_p}\,.$$

Therefore, the number $\tau(6n^2)$ of positive divisors of $6n^2$, i.e., the divisor function, (see, e.g., [3]) is given by

$$\tau(6n^2) = (2\alpha_2 + 2)(2\alpha_3 + 2) \prod_{p \neq 2,3} \left(2\alpha_p + 1\right).$$

The number $N(a, b)$ of all solutions (a, b) of equation (1) is also

$$N(a, b) = (2\alpha_2 + 2)(2\alpha_3 + 2) \prod_{p \neq 2,3} \left(2\alpha_p + 1\right). \qquad (3)$$

EXAMPLE 1.

If $n = 10 = 2 \cdot 5$, then $6n^2 = 600 = 2^3 3^1 5^2$, so the number of positive divisors, and also the number of all solutions of equation (1), is

$$N(a, b) = \tau(600) = 4 \cdot 2 \cdot 3 = 24\,.$$

These divisors x of 600 and the corresponding quotients $y = 600/x$ are listed in Table 1.

Table 1 Twenty-four positive divisors and quotients of the number $6n^2 = 600$

i	j	k	$2^i3^j5^k$	divisor x	quotient y		i	j	k	$2^i3^j5^k$	divisor x	quotient y
0	0	0	$2^03^05^0$	1	600		2	0	0	$2^23^05^0$	4	150
0	0	1	$2^03^05^1$	5	120		2	0	1	$2^23^05^1$	20	30
0	0	2	$2^03^05^2$	25	24		2	0	2	$2^23^05^2$	100	6
0	1	0	$2^03^15^0$	3	200		2	1	0	$2^23^15^0$	12	50
0	1	1	$2^03^15^1$	15	40		2	1	1	$2^23^15^1$	60	10
0	1	2	$2^03^15^2$	75	8		2	1	2	$2^23^15^2$	300	2
1	0	0	$2^13^05^0$	2	300		3	0	0	$2^33^05^0$	8	75
1	0	1	$2^13^05^1$	10	60		3	0	1	$2^33^05^1$	40	15
1	0	2	$2^13^05^2$	50	12		3	0	2	$2^33^05^2$	200	3
1	1	0	$2^13^15^0$	6	100		3	1	0	$2^33^15^0$	24	25
1	1	1	$2^13^15^1$	30	20		3	1	1	$2^33^15^1$	120	5
1	1	2	$2^13^15^2$	150	4		3	1	2	$2^33^15^2$	600	1

Source: own computations in Maple 2025

Since $a = x + 2n = x + 20$, $b = y + 3n = y + 30$, we obtain the following Table 2.

Table 2 Twenty-four positive pairs (a, b) satisfying equation (1) for $n = 10$

divisor x	quotient y	numerator a	numerator b		divisor x	quotient y	numerator a	numerator b
1	600	21	630		25	24	45	54
2	300	22	330		30	20	50	50
3	200	23	230		40	15	60	45
4	150	24	180		50	12	70	42
5	120	25	150		60	10	80	40
6	100	26	130		75	8	95	38
8	75	28	105		100	6	120	36
10	60	30	90		120	5	140	35
12	50	32	80		150	4	170	34
15	40	35	70		200	3	220	33
20	30	40	60		300	2	320	32
24	25	44	55		600	1	620	31

Source: own computations in Maple 2025

Difference case

We now derive the number of decompositions of the form (2), that is,

$$\frac{1}{n}=\frac{2}{c}-\frac{3}{d},$$

where c,d are positive integers with $c<d$. Multiplying both sides by ncd, we obtain

$$cd=n(2d-3c)\,,$$

that is,

$$-cd+2nd-3nc=0\,.$$

Adding $6n^2$ to both sides and factoring (by applying SFFT), we get

$$(2n-c)(d+3n)=6n^2.$$

Let us denote

$$u=2n-c>0\,,\qquad v=d+3n>0\,.$$

Then

$$uv=6n^2,\qquad c=2n-u\,,\qquad d=v-3n\,.$$

Since c,d are positive integers, we obtain the inequalities

$$2n-u>0\,,\qquad v-3n>0\,.$$

Thus, the divisors u of $6n^2$ must satisfy

$$u<2n\,.$$

It follows that

$$v=\frac{6n^2}{u}>\frac{6n^2}{2n}=3n\geq 3n+1\,,$$

and therefore, all solutions (c,d) of equation (2) correspond to divisors u of $6n^2$ such that

$$1\leq u<2n\,.$$

Since it holds $2n<\sqrt{6}n$ for all positive integers n, we obtain

$$2n<\sqrt{6n^2}\,,$$

so for the number $\tau(6n^2)$ of positive divisors of $6n^2$, i.e., for the number $N(c,d)$ of all solutions (c,d) of equation (2), we obtain it holds

$$N(c,d)\leq\left\lfloor\frac{1}{2}\tau(6n^2)\right\rfloor\,, \tag{4}$$

where $\lfloor\cdot\rfloor$ denotes the *floor function*. The exact amount $N(c,d)$ of all solutions (c,d) is given by the number of divisors u of $6n^2$ with $u<2n$.

EXAMPLE 2.

If $n = 10 = 2 \cdot 5$, then $6n^2 = 600 = 2^3 3^1 5^2$, so the number of positive divisors and thus the number of all solutions of equation (2) is, by Example 1,

$$N(c,d) \leq \left\lfloor \frac{1}{2}\tau(600) \right\rfloor = \left\lfloor \frac{4 \cdot 2 \cdot 3}{2} \right\rfloor = \lfloor 12 \rfloor = 12\,.$$

These divisors $1 \leq u < 20$ and the corresponding quotients $v = 600/u$ are listed in Table 3.

Table 3 Ten positive divisors less than 20 and quotients of the number $6n^2 = 600$

i	j	k	$2^i 3^j 5^k$	divisor u	quotient v		i	j	k	$2^i 3^j 5^k$	divisor u	quotient v
0	0	0	$2^0 3^0 5^0$	1	600		1	0	1	$2^1 3^0 5^1$	10	60
0	0	1	$2^0 3^0 5^1$	5	120		1	1	0	$2^1 3^1 5^0$	6	100
0	1	0	$2^0 3^1 5^0$	3	200		2	0	0	$2^2 3^0 5^0$	4	150
0	1	1	$2^0 3^1 5^1$	15	40		2	1	0	$2^2 3^1 5^0$	12	50
1	0	0	$2^1 3^0 5^0$	2	300		3	0	0	$2^3 3^0 5^0$	8	75

Source: own computations in Maple 2025

Since $c = 2n - u = 20 - u$, $d = v - 3n = v - 30$, we obtain Table 4 with numerators c and d, where u is arranged in ascending order and simultaneously c in descending order.

Table 4 Ten positive pairs (c, d) satisfying equation (2) for $n = 10$

divisor u	quotient v	numerator c	numerator d		divisor u	quotient v	numerator c	numerator d
1	600	19	570		6	100	14	70
2	300	18	270		8	75	12	45
3	200	17	170		10	60	10	30
4	150	16	120		12	50	8	20
5	120	15	90		15	40	5	10

Source: own computations in Maple 2025

NUMERICAL VERIFICATION

We solve the problem of determining the decompositions of a reciprocal into sums and differences of reciprocals of the forms $2/a$ and $3/b$. For this purpose, we employ two procedures, `sumdecomp` and `diffdecomp`, which determine the total number of sum and difference decompositions, respectively, and explicitly list all of them. As an illustration, we performed these calculations for the case $n = 10$.

```
> sumdecomp := proc(n::posint)
     local m, divs, x, y, a, b, sols;
     m := 6*n^2;
     divs := numtheory[divisors](m);
     sols := [];
     for x in divs do
         y := m/x;
         a := x + 2*n;
         b := y + 3*n;
         sols := [op(sols), [a,b]];
     od;
     printf("Number of sum decompositions for n=%d: %d\n", n, nops(sols));
     printf("Solutions (a,b): %a\n", sols);
     return sols;
  end proc:

> diffdecomp := proc(n::posint)
     local m, divs, u, v, c, d, sols;
     m := 6*n^2;
     divs := numtheory[divisors](m);
     sols := [];
     for u in divs do
         if u < 2*n then
            v := m/u;
            c := 2*n - u;
            d := v - 3*n;
            if c > 0 and d > 0 then
                sols := [op(sols), [c,d]];
            fi;
         fi;
     od;
     printf("Number of difference decompositions for n=%d: %d\n", n, nops(sols));
     printf("Solutions (c,d): %a\n", sols);
     return sols;
  end proc:
sumdecomp(10);
diffdecomp(10);
```

```
Number of sum decompositions for n=10: 24
Solutions (a,b): [[21, 630], [22, 330], [23, 230], [24, 180], [25,
150], [26, 130], [28, 105], [30, 90], [32, 80], [35, 70], [40, 60],
[44, 55], [45, 54], [50, 50], [60, 45], [70, 42], [80, 40], [95,
38], [120, 36], [140, 35], [170, 34], [220, 33], [320, 32], [620,
31]]

Number of difference decompositions for n=10: 10
Solutions (c,d): [[19, 570], [18, 270], [17, 170], [16, 120], [15,
90], [14, 70], [12, 45], [10, 30], [8, 20], [5, 10]]
```

The outputs give the total number of sum and difference decompositions for $n = 10$, in agreement with the result obtained in Examples 1 and 2 and confirm the theoretical formulas (3) and (4).

CONCLUSIONS

In this paper we have investigated the decompositions of the reciprocal $1/n$ of a positive integer n into the sums and the differences of reciprocals of the forms $2/a$ and $3/b$. We determined that the total number of all such decompositions, using Simon's Favorite Factoring Trick, is given by the number of positive divisors of $6n^2$. We derived that the number of sum decompositions of the form $2/a + 3/b$ is given by the formula

$$N(a,b) = (2\alpha_2 + 2)(2\alpha_3 + 2) \prod_{p \neq 2,3} (2\alpha_p + 1)$$

and that the number of difference decompositions of the form $2/c - 3/d$ is given by inequality

$$N(c,d) \leq \left\lfloor \frac{1}{2} \tau(6n^2) \right\rfloor,$$

where the exact amount $N(c,d)$ of all solutions (c,d) is given by the number of divisors u of $6n^2$ with $u < 2n$.

These decomposition results were illustrated for the number $n = 10$.
Finally, we verified these results using the basic programming features of the computer algebra system Maple 2025, where we explicitly determined the decompositions for $n = 10$.

In conclusion, it can be stated that the decompositions of a positive integer reciprocal into sums and differences of the form $2/a$ and $3/b$ belong to the relatively simple, yet at the same time interesting parts of higher mathematics.

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CONFLICT OF INTEREST

The author declares no conflict of interests or competing interests.

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