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Original Paper

Complex numbers with new definitions of imaginary base numbers

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ABSTRACT

A systematic way of producing new imaginary base numbers is outlined. The new imaginary bases are distinguished from the usual imaginary number *i* by employing another symbol *j*. The relationships of the new base numbers with the usual one are discussed. Several different imaginary base numbers are derived. Complex roots of algebraic equations can be expressed in terms of the new base numbers also. Basic arithmetic operations can be conducted in the new base system by using the definitions. Some potential application areas of the new base systems are also discussed.

KEYWORDS: imaginary number, roots of functions, basic operations

JEL CLASSIFICATION: C20, C60

INTRODUCTION

Although the existence of complex numbers in the real world is questionable, they have proven to be quite useful in handling real world problems. In some problems of nonlinear dynamics, a resort to complex analysis and then returning back to real quantities reduces the algebra much and eliminates the need of memorizing complicated trigonometrical relationships [3], [4]. The fundamental mathematical physics problems such as the wave equation and heat equation can easily be handled using complex analysis [6]. Schrödinger equation, a fundamental equation in quantum mechanics employs complex numbers and functions [8], [9]. Two-dimensional flows can be expressed in the complex plane if the flow is inviscid. A curvilinear coordinate system in the complex plane is proposed by Kaplun [2] in which the coordinates are the streamlines and velocity potentials. This coordinate system is used to express the flow of a non-Newtonian fluid [7]. Debnath [1] gave a brief history of some remarkable numbers including the imaginary

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number. Historical roots as well as application areas are discussed for complex numbers by Wu and Yuan [11].

The imaginary number definition stemmed from the well-known simple algebraic equation $x^2 = -1$ which does not have a real solution. Hence the solution is assumed to be imaginary with the symbol $i = \sqrt{-1}$. Geometrically speaking, $y = x^2$is always positive, or zero at $x = 0$, in the two-dimensional real space but the constant function $y = -1$ is negative in the whole domain. Therefore, there is no intersection point or real solution to the simple algebraic equation which led to the definition of the base imaginary number i. However, there are infinitely many algebraic equations which do not have real solutions and the idea of defining the roots of the more general equation $f(x) = -1$ with $f(x) \geq 0$ in the interval $(-\infty, +\infty)$ leaded to different imaginary base number definitions. In this work, first the general definition for such base numbers is given. Then following the general definition, several new imaginary base numbers are proposed. The relationships of the new base numbers with the usual base number are derived. Basic arithmetic operations are given for the new base imaginary numbers.

One immediate advantage of the new bases is that the complex solutions of some algebraic equations as outlined in this study can be expressed in simpler forms in the new bases. This feature can also be employed in expressing solutions of differential equations in simpler forms and a possible application from mechanical vibrations is given at the end. Another important application is the mapping of shapes into other forms which is used in designing objects. A mapping from circle to ellipse is given in the last section. The topic considered is quite new and potential applications of such new complex number bases remain an open question of investigation. However, considering the vast applications of the classical complex number bases in mathematical physics and geometry, it may be estimated that the new base definitions might also have some interesting applications in physical sciences and geometry.

MATERIAL AND METHODS

A nonzero arbitrary function $y = f(x)$ and the constant line $y = -1$ are shown in Figure 1. Obviously, the two functions do not intersect and there is no real solution to the algebraic problem $f(x) = -1$.

The new base imaginary number definitions follow from this more general algebraic equation.

Definition 1. For the arbitrary function $f(x) \geq 0$ over the whole domain $(-\infty, +\infty)$, the root j of $f(x) = -1$ determines a new imaginary base number, i.e. $j = f^{-1}(-1)$ □

Note that the inverse function swaps domain and range. In other words, the domain of $f^{-1}(x)$ is the same as the range of $f(x)$. The range of $f^{-1}(x)$ is the same as the domain of $f(x)$. However, this rule is violated in Definition 1 which leads to a new definition of imaginary base number.

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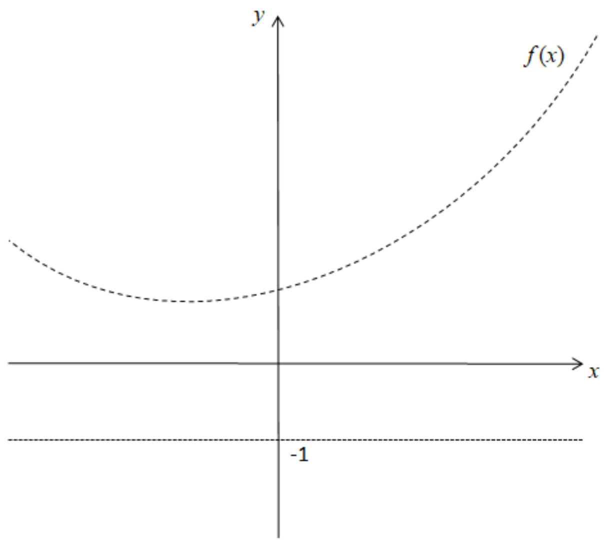
Figure 1 The schematic representation of the algebraic equation $f(x) = -1$



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Theorem 1. The imaginary base number defined in definition 1 is related to the usual base number $i = \sqrt{-1}$ by $j = a + bi$ where $a \in \mathbb{R}, b \in \mathbb{R} - \{0\}$ □

Proof 1. Since $f(x) \geq 0$ over the whole domain, the algebraic equation $f(x) = -1$ does not possess a pure real solution which indicates that $b \neq 0$. Furthermore $j = f^{-1}(-1)$ or $j = f^{-1}(e^{i\pi})$ from the famous Euler equation. Therefore, j is definitely a complex valued number, which may have a zero or nonzero real part □

RESULTS AND DISCUSSION

Examples of new base numbers, basic arithmetic operations and some potential applications areas will be given in this section.

New Base Numbers

Depending on the theory given, new imaginary base numbers are derived in this section.

Example 1. Consider $f(x) = e^x$. The algebraic equation $e^x = -1$ does not possess a real solution. Assume that the root of this equation is j. Hence $e^j = -1$ or

$$j = \ln(-1). \tag{1}$$

Since the natural logarithm is undefined for negative values, this expression is indeed an imaginary number and can be considered as a new imaginary base number. While e^x is defined over the entire real axis, the inverse function $\ln x$ is defined only in the positive real axis. This is the exact reason of equation (1) ending up as an imaginary number.

To find the relationship of this new base number with the usual one, note that

$$e^{j} = -1 = e^{i\pi} \; . \tag{2}$$

Hence,

$$j = \pi i \; . \tag{3}$$

As an example, consider the equation $e^{x} = -e$. To find the root in the new base, take the natural logarithm of both sides, $x = \ln(-e) = \ln(e) + \ln(-1) = 1 + j$ which is a simpler solution compared to the usual base solution which is $x = 1 + \pi i$.

Example 2. Consider the algebraic equation $f(x) = \frac{1}{1+e^{-x}} = -1$. Solving for e^{-x}, the equation is $e^{-x} = -2$ or if the root is considered to be the new imaginary base number, then $e^{-j} = -2$. Solving for j

$$j = -\ln(-2) = -\ln(2) - \ln(-1), \tag{4}$$

or the relationship with the usual basis is

$$j = -\ln(2) - \pi i \; . \tag{5}$$

Example 3. Consider the algebraic equation $f(x) = x^{2n} = -1$ where $n \in \mathbb{Z}^{+}$. Then the root can be considered as a new imaginary base number which is

$$j = (-1)^{1/2n} \; . \tag{6}$$

To find the relationship with the usual base number, the Euler formula is employed

$$j = i^{1/n} = e^{i\frac{\pi}{2n}} = \cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right) . \tag{7}$$

As stated in Theorem 1, the new base numbers can always be represented in the form $j = a + bi$ in terms of the usual base. Note that Euler formula can only be used for the old base number and is not valid for the new bases.

Example 4. Consider the algebraic equation $f(x) = \sin x + 1 = -1$ where $f(x)$ is definitely non-negative in the whole domain of x. Then the root can be considered as a new imaginary base number which is

$$j = \mathrm{Arcsin}(-2) \; . \tag{8}$$

To find the relationship with the usual base number,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = -2 \; , \tag{9}$$

or

$$e^{ix} - e^{-ix} = -4i \; . \tag{10}$$

If $u = e^{ix}$, then

$$u - \frac{1}{u} = -4i\ . \tag{11}$$

Multiplying by u and rearranging

$$u^2 + 4iu - 1 = 0\ , \tag{12}$$

with a solution $u = \left(-2 \mp \sqrt{3}\right)i$. Hence, if the positive sign solution is taken

$$e^{ix} = \left(-2 + \sqrt{3}\right)i\ . \tag{13}$$

Solving for x, $j = -i\ln\left[\left(-2 + \sqrt{3}\right)i\right]$. To express the solution in a more convenient way, note that for a complex number z, $\ln(z) = ln|z| + i\arg(z)$ [5].

Hence $\ln\left[\left(-2 + \sqrt{3}\right)i\right] = \ln\left(2 - \sqrt{3}\right) + i\frac{3}{2}\pi$. The new base number in terms of the usual base is then

$$j = \frac{3}{2}\pi - i\ln\left(2 - \sqrt{3}\right)\ . \tag{14}$$

Example 5. Consider the algebraic equation $f(x) = \cosh x = -1$. Then the new base number with respect to hyperbolic cosine function would be

$$j = \operatorname{Arccosh}(-1)\ . \tag{15}$$

To find the equivalent in the usual base system,

$$\cosh x = \frac{e^x + e^{-x}}{2} = -1, \tag{16}$$

yielding

$$e^x = -1\ . \tag{17}$$

But this case was already treated in Example 1. Hence,

$$j = \pi i\ . \tag{18}$$

The powers of j can be calculated via the below relationships

$$j^{4k} = \pi^{4k},\ \ j^{4k+1} = \pi^{4k+1}j,\ \ j^{4k+2} = -\pi^{4k+2},\ \ j^{4k+3} = -\pi^{4k+2}j,\quad k = 0,1,2,\ldots \tag{19}$$

Example 6. Consider the general quadratic equation $f(x) = a(x - r)^2 + k = -1$ with $a > 0, k \geq 0$. Then one of the new base numbers with respect to this function is

$$j = r + \sqrt{-\frac{1+k}{a}}\,. \tag{20}$$

The equivalent in the usual base system is therefore,

$$j = r + i\sqrt{\frac{1+k}{a}} . \tag{21}$$

Basic Arithmetic Operations

If the complex numbers are given in one of the new base system

$$w_1 = a_1 + b_1 j, \quad w_2 = a_2 + b_2 j , \tag{22}$$

then the addition and subtraction processes are straightforward yielding

$$w_1 \mp w_2 = a_1 \mp a_2 + (b_1 \mp b_2) j . \tag{23}$$

The multiplication of the two complex numbers will be

$$w_1 w_2 = a_1 a_2 + b_1 b_2 j^2 + (a_1 b_2 + b_1 a_2) j , \tag{24}$$

which requires the specific definition of the new base for calculating the term j^2. If for example $j = \ln(-1) = \pi i$, then $j^2 = -\pi^2$ and the $b_1 b_2 j^2$ contributes only to the real part with $w_1 w_2 = a_1 a_2 - b_1 b_2 \pi^2 + (a_1 b_2 + b_1 a_2) j$. However, for the more general cases of $j = a + bi$ form, the $b_1 b_2 j^2$ term has contributions both to the real and imaginary parts. For division, multiplying by complex conjugates of the denominator works only if the new base is of the form $j = bi$. In that case

$$\frac{w_1}{w_2} = \frac{a_1 + b_1 j}{a_2 + b_2 j} = \frac{a_1 + b_1 j}{a_2 + b_2 j} \frac{a_2 - b_2 j}{a_2 - b_2 j} = \frac{a_1 a_2 - b_1 b_2 j^2 + (b_1 a_2 - a_1 b_2) j}{a_2^2 - b_2^2 j^2} , \tag{25}$$

and the denominator is ensured to be real. For $j = \ln(-1) = \pi i$, the result is

$$\frac{w_1}{w_2} = \frac{a_1 + b_1 j}{a_2 + b_2 j} = \frac{a_1 a_2 + b_1 b_2 \pi^2 + (b_1 a_2 - a_1 b_2) j}{a_2^2 + b_2^2 \pi^2} . \tag{26}$$

If j is in the form $j = a + bi$, then instead of the complex conjugate, the numerator and denominator should be multiplied by $a_2 - c_2 j$ to make the denominator real where

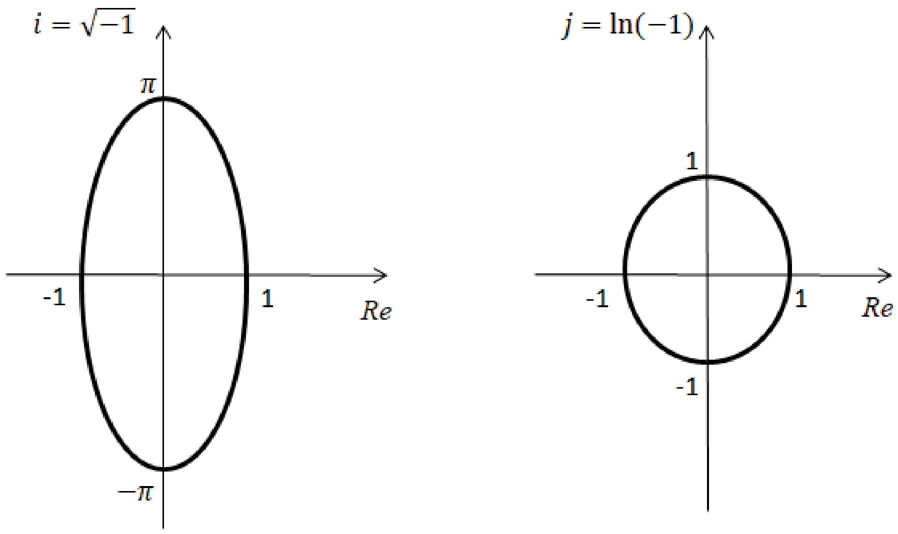
$$c_2 = \frac{a_2 b_2}{a_2 + 2 b_2 a} . \tag{27}$$

If j is in the form $j = bi$, then $a = 0$, and $c_2 = b_2$ reducing to the multiplication by the complex conjugate. The new base system is closed under the basic arithmetic operations.

Potential Applications

It is well known that geometrical objects can be expressed in the complex plane and can be drawn by rotating a complex vector and marking the end point as the trajectory. As such, circles, ellipses and hyperboles can be drawn in the complex plane. Mapping of figures from one shape to another is a practical problem possessing many applications in designing objects. Changes in the basis of the complex numbers can be viewed as mappings from one shape to another. A simple example is given here. A unit circle in the new complex plane (real, $j=\ln(-1)$) corresponds to an ellipse in the standard complex plane (real, $i=\sqrt{-1}$) (Figure 2). Therefore, the base change corresponds to a mapping from the circle to ellipse and vice versa.

Figure 2 Mapping of circle to ellipse



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The base changes may also be useful in expressing the solutions of specific problems in a more simplified form. Complex analysis and representations of solutions in the complex plane are frequently used in vibration problems. For example, for the lightly damped vibrations, the mathematical model is [10]

$$\ddot{x}+2\zeta\omega\dot{x}+\omega^2x=0\,, \tag{28}$$

where $x(t)$ is the vibrational response, ω is the natural frequency and ζ is the dimensionless damping coefficient. It is well-known that the differential equation admits solutions of the form

$$x(t)=Ce^{\left(-\zeta\omega+i\omega\sqrt{1-\zeta^2}\right)t}\,. \tag{29}$$

If one defines a new complex number basis

$$j=-\zeta\omega+i\omega\sqrt{1-\zeta^2}\,, \tag{30}$$

then the solution in this new base would be much simplified

$$x(t)=Ce^{jt}\,. \tag{31}$$

CONCLUSIONS

By generalizing the basic quadratic equation defining the base imaginary number i with employment of arbitrary functions, new imaginary base numbers are defined. The relationship of the new base numbers with the old base numbers turns out to be in the form of a complex number with zero or non-zero real parts. Several new imaginary base numbers are proposed in the worked examples. The basic arithmetic operations are also discussed for the new imaginary base systems.

To the best of the authors' knowledge, this way of defining new complex base numbers did not appear in the literature. In view of the vast applications of complex numbers in applied mathematics and geometry, it is hoped that these introductory definitions and concept might be further exploited to find many new application areas. From the educational point of view, the students may gain better understanding of the bases of complex numbers and their extensions.

CONFLICT OF INTEREST
The author declares no conflict of interests or competing interests.

REFERENCES

[1] Debnath, L. (2015). A brief history of the most remarkable numbers i, e and γ in mathematical sciences with applications. *International Journal of Mathematical Education in Science and Technology*, 46(6), 853-878. Doi: https://doi.org/10.1080/0020739X.2015.1015266

[2] Kaplun, S. (1954). The role of coordinate systems in boundary-layer theory. *Zeitschrift für Angewandte Mathematik und Physik*, 5, 111-135. Doi: https://doi.org/10.1007/BF01600771

[3] Nayfeh, A. H. (1981). *Introduction to Perturbation Techniques.* New York: John Wiley and Sons.

[4] Nayfeh, A. H, & Mook, D. T. (1979). *Nonlinear Oscillations*. New York: John Wiley and Sons.

[5] O'Neil, P. V. (1991). *Advanced Engineering Mathematics*. Belmont, California: Wadsworth.

[6] Pakdemirli, M. (2024). Complex exponential method for solving partial differential equations. *Engineering Transactions*, 72(4), 461-474. Doi: https://doi.org/10.24423/EngTrans.3334.2024

[7] Pakdemirli, M., & Şuhubi, E. S. (1992). Boundary layer theory for second order fluids. *International Journal of Engineering Science*, 30(4), 523-532. Doi: https://doi.org/10.1016/0020-7225(92)90042-F

[8] Reed, B. C. (2022). *Schrödinger's Equation, In: Quantum Mechanics.* Cham: Springer.

[9] Tahir, M. B., Sagir, M., Khan, M. I., & Rafique, M. (2024). *Schrödinger Equation: Fundamentals Aspects and Potential Applications*. BoD–Books on Demand.

[10] Tse, F. S., Morse, I. E., & Hinkle, R. T. (1978). *Mechanical Vibrations, Theory and Applications*. Boston: Allyn and Bacon Inc.

[11] Wu, W., & Yuan, X. (2023). The theoretical significance and reality of imaginary number. *Natural Science*, 15(11), 285-288.