{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2024(10) 2

Received: 2024-09-20
Accepted: 2024-11-29
Online published: 2024-12-31
DOI: **https://doi.org/10.15414/meraa.2024.10.02.34-47**

Original Paper

Problems on the sum of the areas of infinitely many decreasing basic planar shapes

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ABSTRACT

This contribution deals with the calculations of the sums of the areas of infinitely many uniformly decreasing basic planar shapes. Using the formula for the sum of an infinite geometric series, the formulas for the sums of the areas enclosed by an equilateral, isosceles and right triangle, for a square, rectangle, regular hexagon, regular octagon, regular n-sided polygon, circle and ellipse are derived. Computer algebra system Maple was used for numerical calculations.

KEYWORDS: sum of the convergent geometric series, area, triangle, square, rectangle, regular n-sided polygon, circle, ellipse, computer algebra system Maple, programming language MetaPost

JEL CLASSIFICATION: C60, I20

INTRODUCTION

Let us recall some used basic terms concerning geometric series and formulas for areas of the simple shapes. The geometric series is the sum of an infinite number of terms that have a constant ratio between successive terms. For example, the series

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$$

is geometric, because each successive term can be obtained by multiplying the previous term by $1/3$. In general, a geometric series is written as

$$a_1+a_1q+a_1q^2+a_1q^3+\cdots,$$

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where a_1 is the first term or the coefficient of each term and q is the common ratio between adjacent terms.

As the number of terms approaches infinity, the absolute value of q must be less than one for the series to converge. The sum is then given by the formula

$$s = \sum_{k=0}^{\infty} a_1 q^k = \frac{a_1}{1-q}\ . \tag{1}$$

The area enclosed by an **equilateral triangle** with a side of the length a is

$$A_{et} = \frac{\sqrt{3}}{4} a^2. \tag{2}$$

The area enclosed by an **isosceles triangle** with legs of the length a and a base of the length b is

$$A_{it} = \frac{b}{4}\sqrt{4a^2 - b^2}. \tag{3}$$

The area enclosed by a **right triangle** with legs of the lengths a and b is

$$A_{rt} = \frac{ab}{2}. \tag{4}$$

The area enclosed by a **square** with a side of the length a is

$$A_{sq} = a^2. \tag{5}$$

The area enclosed by a **rectangle** with sides of the lengths a and b is

$$A_{re} = ab. \tag{6}$$

The area enclosed by a **regular hexagon** with a side of the length a is

$$A_{rh} = \frac{3}{2}\sqrt{3}a^2. \tag{7}$$

The area enclosed by a **regular octagon** with a side of the length a is

$$A_{ro} = 2\left(1 + \sqrt{2}\right)a^2. \tag{8}$$

The area enclosed by a **regular n-sided polygon** with a side of the length a is

$$A_{np} = \frac{1}{4} n a^2 \cot\left(\frac{\pi}{n}\right). \tag{9}$$

The area enclosed by a **circle** with a radius of the length a is

$$A_{ci} = \pi a^2. \tag{10}$$

The area enclosed by an **ellipse** with semi-axes of the lengths a and b is

$$A_{el} = \pi ab. \tag{11}$$

Basic knowledge regarding infinite geometric series can be found, among others, on Wikipedia, on the website [18]. The above-mentioned formulas (2) – (11) can also be found also on Wikipedia on the website [17]. An interesting article discussing different possibilities of adding infinite series is the article [9] by Dan Kalman. The sum of a special geometric series is dealt with, for example, in article [5]. In addition to the basic problems, which relate to the sum of the areas of shrinking regions formed by one planar geometric shape, it is of course possible to create problems in which different geometric shapes alternate. This situation is described, for example, in contribution [11].

Example: Determine the sum of infinitely many areas, which are formed by inscribing a circle in a square with a side of length a then inscribing a square in it, a circle in it again, etc.

Solution: It is obvious that the sum s of these areas is the sum of the areas s_1 of infinitely many squares and the areas s_2 of infinitely many circles. At the same time, according to Figure 1, the contents of the squares form a geometric sequence with the first term $a_1 = a^2$ and the common ratio $q_1 = 0.5$, and the contents of the circles form a geometric sequence with the first term $b_1 = \pi(0.5a)^2$ and the common ratio $r_1 = 0.5$. Therefore, we get

$$s = \sum_{k=0}^{\infty} a_1 {q_1}^k + \sum_{k=0}^{\infty} b_1 {r_1}^k = \frac{a^2}{1-0.5} + \frac{\pi a^2}{4(1-0.5)} = \frac{a^2}{0.5}\left(1+\frac{\pi}{4}\right) = \frac{(\pi+4)a^2}{2}.$$

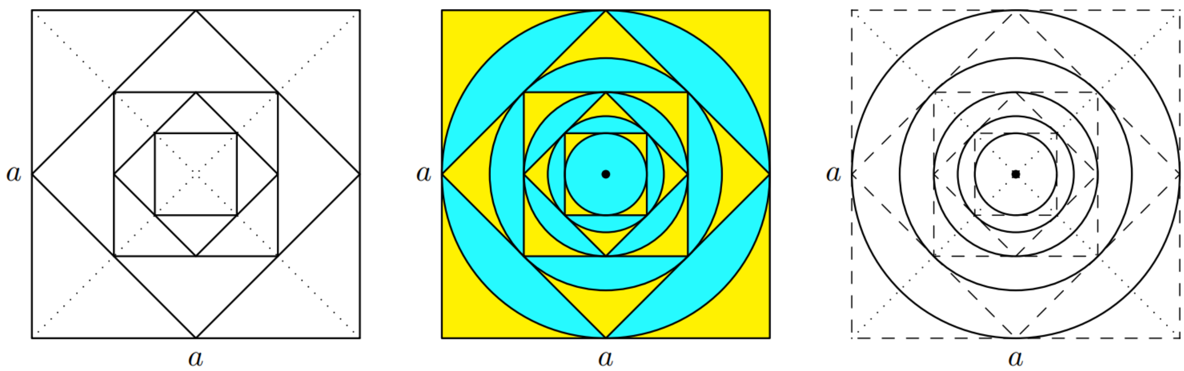

Figure 1 The sum s of the areas enclosed by the sequence of decreasing squares and disks
Source: own modelling in MetaPost

A similar task can be found, for example, in the study material [3], which contains examples of geometric sequences and infinite geometric series. Another study material dealing with infinite geometric series is, for example, the material [4]. Examples of infinite geometric series are included in the educational material [1].

Planar shapes are dealt with in article [7], and the contribution [2] contains examples of the content of planar shapes. Filling part of a planar area with planar shapes is the subject of a number of other articles. Article [8] deals with a sequence of squares whose contents form a harmonic series. Article [6] deals with filling part of a planar area with triangles and squares. The contribution [14] deals with the sum of the areas of the circles between two semicircles. Filling a square with specific circles is the topic of student's work [10]. Tasks dealing with the contents of the surfaces of inscribed planar shapes are solved on YouTube, for example, in posts [12], [13], [15] and [16].

Problems similar to the problems in this article and solved using formula (1) could also be formulated for calculating the sum of the volumes of shrinking bodies.

THE SUM OF THE AREAS OF INFINITELY MANY DECREASING BASIC PLANAR SHAPES

Let us consequently consider ten basic planar shapes – the equilateral, isosceles and right triangle, square, rectangle, regular hexagon, regular octagon, regular n-sided polygon, circle and ellipse. By means of formula (1), we derive the sums of the areas enclosed by these infinitely many decreasing shapes depending on a coefficient $c \in (0,1)$.

Simplified pictures created in the programming language MetaPost with six first decreasing shapes will illustrate all these ten cases for the reduction coefficient $c = 0.5$.

1. Equilateral triangle

The sum $s_{eq}(a;c)$ of the areas enclosed by infinitely many decreasing equilateral triangles with the first area $A_{et} = \frac{\sqrt{3}}{4}a^2$, with the reduction coefficient $c \in (0,1)$ and with the side of the length a according the formulas (1) and (2) is

$$s_{et}(a;c) = \sum_{k=0}^{\infty} \frac{\sqrt{3}}{4}(ac^k)^2 = \frac{\sqrt{3}}{4}a^2(1 + c^2 + c^4 + \cdots) = \frac{\frac{\sqrt{3}}{4}a^2}{1 - c^2} = \frac{\sqrt{3}a^2}{4(1 - c^2)}\,. \qquad (12)$$

For example, if $a = 1$ and $c = 0.5$, we have the sum of the areas

$$s_{et}(1;0.5) = \frac{\sqrt{3}}{4(1 - 0.25)} = \frac{\sqrt{3}}{3} \doteq 0.577350\,.$$

The following Figure 2 illustrates this situation.

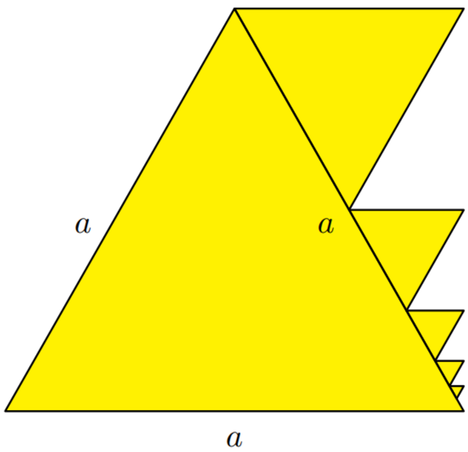

Figure 2 The sum $s_{et}(1;0.5)$ of the areas enclosed by first six decreasing equilateral triangles
Source: own modelling in MetaPost

2. Isosceles triangle

The sum $s_{it}(a, b; c)$ of the areas enclosed by infinitely many decreasing isosceles triangles with the first area $A_{it} = \frac{b}{4}\sqrt{4a^2 - b^2}$, where $2a > b$ with the reduction coefficient $c \in (0,1)$, with a base of the length b and with legs of the length a according the formulas (1) and (3) is

$$s_{it}(a, b; c) = \sum_{k=0}^{\infty} \frac{bc^k}{4}\sqrt{4(ac^k)^2 - (bc^k)^2} = \frac{b}{4}\sqrt{4a^2 - b^2}\sum_{k=0}^{\infty}(c^k)^2 = \frac{b\sqrt{4a^2 - b^2}}{4(1 - c^2)}. \quad (13)$$

For example, if $a = 5$, $b = 8$ and $c = 0.5$, we have the sum of the areas

$$s_{it}(5, 8; 0.5) = \frac{8\sqrt{100 - 64}}{4(1 - 0.25)} = \frac{48}{3} = 16\,.$$

The following Figure 3 illustrates this situation.

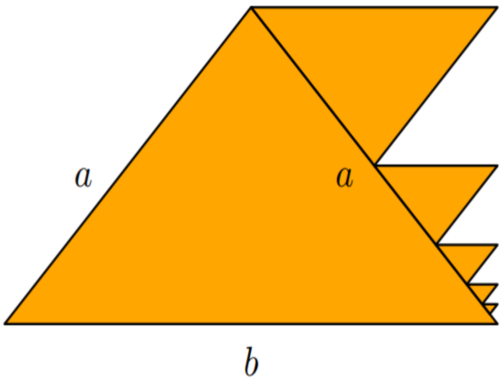

Figure 3 The sum $s_{it}(5, 8; 0.5)$ of the areas enclosed by first six decreasing isosceles triangles
Source: own modelling in MetaPost

3. Right triangle

The sum $s_{rt}(a, b; c)$ of the areas enclosed by infinitely many decreasing right triangles with the first area $A_{rt} = \frac{ab}{2}$, with the reduction coefficient $c \in (0,1)$ and with legs of the lengths a and b the formulas (1) and (4) is

$$s_{rt}(a, b; c) = \sum_{k=0}^{\infty} \frac{(ac^k)(bc^k)}{2} = \frac{ab}{2}(1 + c^2 + c^4 + \cdots) = \frac{ab}{2(1 - c^2)}. \quad (14)$$

For example, if $a = 4$, $b = 3$ and $c = 0.5$, we have the sum of the areas

$$s_{rt}(4,3;0.5) = \frac{12}{2(1-0.25)} = 8\,.$$

The following Figure 4 illustrates this situation.

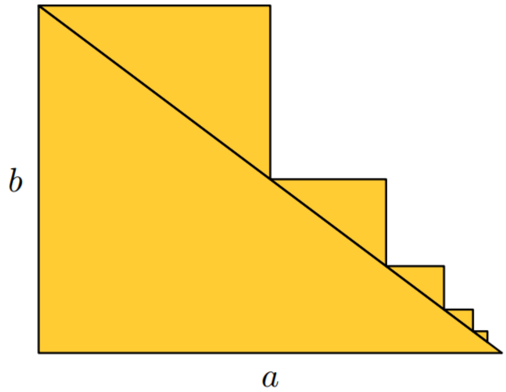

Figure 4 The sum $s_{rt}(4,3;0.5)$ of the areas enclosed by first six decreasing right triangles
Source: own modelling in MetaPost

4. Square

The sum $s_{sq}(a;c)$ of the areas enclosed by infinitely many decreasing squares with the first area $A_{sq} = a^2$, with the reduction coefficient $c \in (0,1)$ and with the side of the length a according the formulas (1) and (5) is

$$s_{sq}(a;c) = \sum_{k=0}^{\infty}(ac^k)^2 = a^2(1 + c^2 + c^4 + \cdots) = \frac{a^2}{1-c^2}\,. \qquad (15)$$

For example, if $a = 1$ and $c = 0.5$, we have the sum of the areas

$$s_{sq}(1;0.5) = \frac{1}{1-0.25} = \frac{4}{3} \doteq 1.333333\,.$$

The following Figure 5 illustrates this situation.

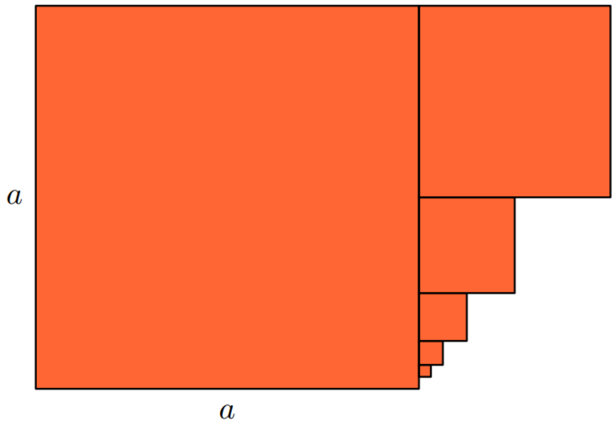

Figure 5 The sum $s_{sq}(1; 0.5)$ of the areas enclosed by first six decreasing squares
Source: own modelling in MetaPost

5. Rectangle

The sum $s_{re}(a, b; c)$ of the areas enclosed by infinitely many decreasing rectangles with the first area $A_{re} = ab$, with the reduction coefficient $c \in (0,1)$ and with sides of the lengths a and b according the formulas (1) and (6) is

$$s_{re}(a, b; c) = \sum_{k=0}^{\infty} (ac^k)(bc^k) = ab(1 + c^2 + c^4 + \cdots) = \frac{ab}{1 - c^2}\,. \quad (16)$$

For example, if $a = 4$, $b = 3$ and $c = 0.5$, we have the sum of the areas

$$s_{re}(4, 3; 0.5) = \frac{12}{1 - 0.25} = 16\,.$$

The following Figure 6 illustrates this situation.

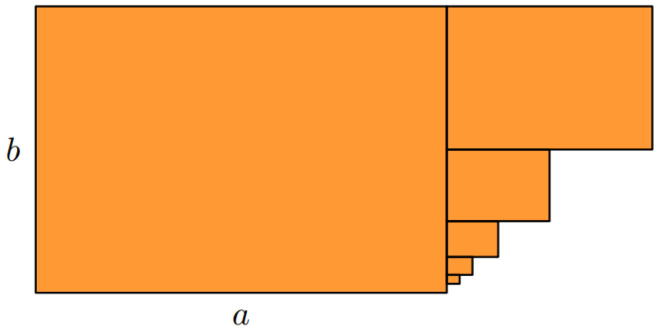

Figure 6 The sum $s_{re}(4, 3; 0.5)$ of the areas enclosed by first six decreasing rectangles
Source: own modelling in MetaPost

6. Regular hexagon

The sum $s_{rh}(a;c)$ of the areas enclosed by infinitely many decreasing regular hexagons with the first area $A_{rh} = \frac{3}{2}\sqrt{3}a^2$, with the reduction coefficient $c \in (0,1)$ and with the side of the length a according the formulas (1) and (7) is

$$s_{rh}(a;c) = \sum_{k=0}^{\infty} \frac{3}{2}\sqrt{3}(ac^k)^2 = \frac{3}{2}\sqrt{3}\,a^2(1 + c^2 + c^4 + \cdots) = \frac{\frac{3}{2}\sqrt{3}a^2}{1 - c^2} = \frac{3\sqrt{3}a^2}{2(1 - c^2)}\,. \quad (17)$$

For example, if $a = 1$ and $c = 0.5$, we have the sum of the areas

$$s_{rh}(1;0.5) = \frac{3\sqrt{3}}{2(1 - 0.25)} = 2\sqrt{3} \doteq 3.464102\,.$$

The following Figure 7 illustrates this situation.

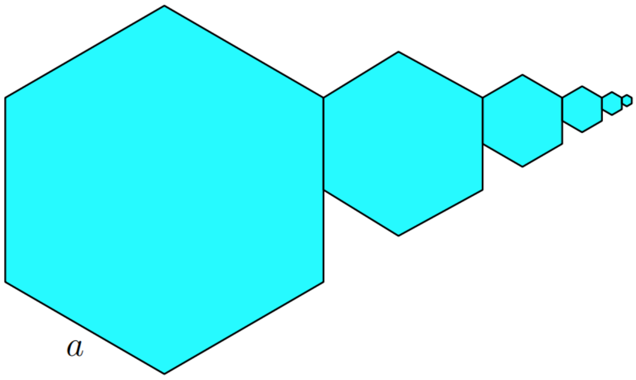

Figure 7 The sum $s_{rh}(1;0.5)$ of the areas enclosed by first six decreasing regular hexagons
Source: own modelling in MetaPost

7. Regular octagon

The sum $s_{ro}(a;c)$ of the areas enclosed by infinitely many decreasing regular octagons with the first area $A_{ro} = 2(1 + \sqrt{2})a^2$, with the reduction coefficient $c \in (0,1)$ and with the side of the length a according the formulas (1) and (8) is

$$s_{ro}(a;c) = \sum_{k=0}^{\infty} 2(1 + \sqrt{2})(ac^k)^2 = 2(1 + \sqrt{2})a^2(1 + c^2 + c^4 + \cdots) = \frac{2(1 + \sqrt{2})a^2}{1 - c^2}. \quad (18)$$

For example, if $a = 1$ and $c = 0.5$, we have the sum of the areas

$$s_{ro}(1;0.5) = \frac{2(1 + \sqrt{2})}{1 - 0.25} = \frac{8(1 + \sqrt{2})}{3} \doteq 6.437903\,.$$

The following Figure 8 illustrates this situation.

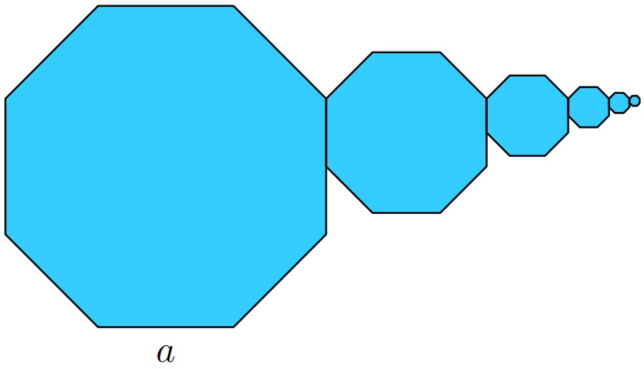

Figure 8 The sum $s_{rh}(1; 0.5)$ of the areas enclosed by first six decreasing regular octagons
Source: own modelling in MetaPost

8. Regular n-sided polygon

The sum $s_{np}(a, n; c)$ of the areas enclosed by infinitely many decreasing regular n-sided polygons with the first area $A_{np} = \frac{1}{4} n a^2 \cot\left(\frac{\pi}{n}\right)$, with the reduction coefficient $c \in (0,1)$ and with the side of the length a according the formulas (1) and (9) is

$$s_{np}(a, n; c) = \sum_{k=0}^{\infty} \frac{1}{4} n (a c^k)^2 \cot\left(\frac{\pi}{n}\right) = \frac{1}{4} n \cot\left(\frac{\pi}{n}\right) a^2 (1 + c^2 + c^4 + \cdots) = \frac{n \cot\left(\frac{\pi}{n}\right) a^2}{4(1 - c^2)} . \tag{19}$$

For example, if $a = 1$, $n = 12$ and $c = 0.5$, we have the sum of the areas

$$s_{ro}(1,12; 0.5) = \frac{12 \cot\left(\frac{\pi}{12}\right)}{4(1 - 0.25)} \doteq \frac{12 \cdot 3.732051}{3} \doteq 14.928203 .$$

Similar situations are illustrated in Figures 7 and 8.

9. Circle

The sum $s_{ci}(a; c)$ of the areas enclosed by infinitely many decreasing circles with the first area $A_{ci} = \pi a^2$, with the reduction coefficient $c \in (0,1)$ and with the radius of the length a according the formulas (1) and (9) is

$$s_{ci}(a; c) = \sum_{k=0}^{\infty} \pi (a c^k)^2 = \pi a^2 (1 + c^2 + c^4 + \cdots) = \frac{\pi a^2}{1 - c^2} . \tag{20}$$

For example, if $a = 1$ and $c = 0.5$, we have the sum of the areas

$$s_{ci}(1; 0.5) = \frac{\pi}{1 - 0.25} = \frac{4\pi}{3} \doteq 4.188790 .$$

The following Figure 9 illustrates this situation.

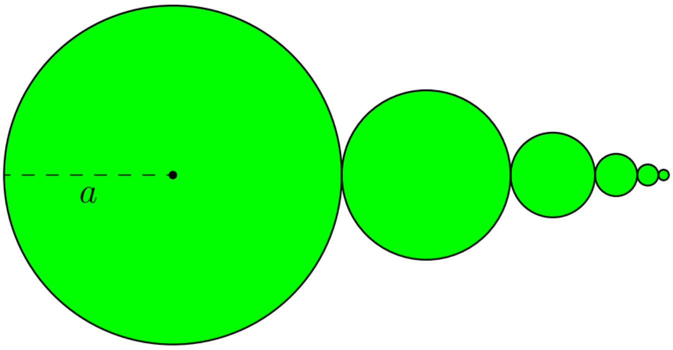

Figure 9 The sum $s_{ci}(1; 0.5)$ of the areas enclosed by first six decreasing circles
Source: own modelling in MetaPost

10. Ellipse

The sum $s_{el}(a, b; c)$ of the areas enclosed by infinitely many decreasing rectangles with the first area $A_{el} = \pi ab$, with the reduction coefficient $c \in (0,1)$ and with semi-axes of the lengths a and b according the formulas (1) and (10) is

$$s_{el}(a, b; c) = \sum_{k=0}^{\infty} \pi(ac^k)(bc^k) = \pi ab(1 + c^2 + c^4 + \cdots) = \frac{\pi ab}{1 - c^2}\,. \qquad (21)$$

For example, if $a = 4$, $b = 3$ and $c = 0.5$, we have the sum of the areas

$$s_{el}(4, 3; 0.5) = \frac{12\pi}{1 - 0.25} = 16\pi\,.$$

The following Figure 10 illustrates this situation.

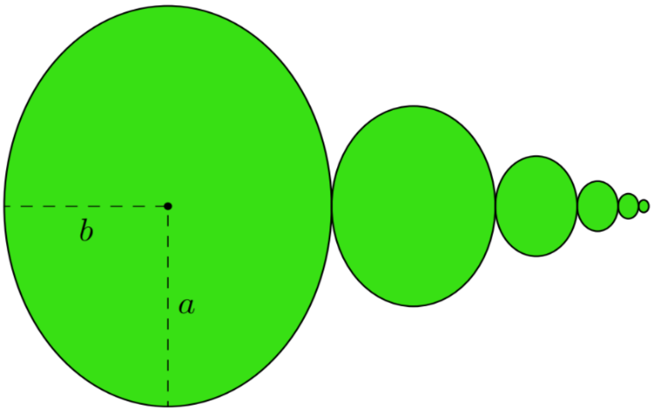

Figure 10 The sum $s_{el}(4, 3; 0.5)$ of the areas enclosed by first six decreasing ellipses
Source: own modelling in MetaPost

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OVERVIEW OF THE RESULTS ALONG WITH SOME OF THEIR SPECIFIC NUMERICAL VALUES

The following Table 1 clearly shows the results derived above for the sum of the areas of infinitely many decreasing areas enclosed by equilateral triangles, isosceles triangles, right triangles and rectangles.
Table 2 similarly provides an overview of the results derived above for the sum of the areas of infinitely many decreasing areas enclosed by squares, regular hexagons, regular octagons, regular n-sided polygons, circles and ellipses.
Tables 1 and 2 also contain some selected numerical results for specific values of the variables a or a and b or a and n and for the reduction coefficient $c = 0.5$. All these results were computed by using the computer algebra system Maple.

Table 1 Some values of the sums $s_{et}(a; 0.5), s_{it}(a, b; 0.5), s_{rt}(a, b; 0.5)$ and $s_{re}(a, b; 0.5)$

equilateral triangles	$s_{et}(a;c) = \dfrac{\sqrt{3}a^2}{4(1-c^2)}$								$s_{et}(a;0.5) = \dfrac{\sqrt{3}a^2}{3}$
	$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$	$a = 6$	$a = 7$	$a = 8$	$a = 9$
$s_{et}(a;0.5)$	$\sqrt{3}/3$	$4\sqrt{3}/3$	$3\sqrt{3}$	$16\sqrt{3}/3$	$25\sqrt{3}/3$	$12\sqrt{3}$	$49\sqrt{3}/3$	$64\sqrt{3}/3$	$27\sqrt{3}$
isosceles triangles	$s_{it}(a,b;c) = \dfrac{b\sqrt{4a^2-b^2}}{4(1-c^2)}$								$s_{it}(a,b;0.5) = \dfrac{b\sqrt{4a^2-b^2}}{3}$
$s_{it}(a,b;0.5)$	$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$	$a = 6$	$a = 7$	$a = 8$	$a = 9$
$b = 1$	$\sqrt{3}/3$	$\sqrt{15}/3$	$\sqrt{35}/3$	$3\sqrt{7}/3$	$\sqrt{11}$	$\sqrt{143}/3$	$\sqrt{195}/3$	$\sqrt{255}/3$	$\sqrt{323}/3$
$b = 2$	×	$4\sqrt{3}/3$	$8\sqrt{2}/3$	$4\sqrt{15}/3$	$8\sqrt{6}/3$	$4\sqrt{35}/3$	$16\sqrt{3}/3$	$12\sqrt{7}/3$	$16\sqrt{5}/3$
$b = 3$	×	$\sqrt{7}$	$3\sqrt{3}$	$\sqrt{55}$	$\sqrt{91}$	$3\sqrt{15}$	$\sqrt{187}$	$\sqrt{247}$	$3\sqrt{35}$
$b = 4$	×	×	$8\sqrt{5}/3$	$16\sqrt{3}/3$	$8\sqrt{21}/3$	$32\sqrt{2}/3$	$24\sqrt{5}/3$	$16\sqrt{15}/3$	$8\sqrt{77}/3$
$b = 5$	×	×	$5\sqrt{11}/3$	$5\sqrt{39}/3$	$25\sqrt{5}/3$	$5\sqrt{119}/3$	$5\sqrt{19}$	$5\sqrt{231}/3$	$5\sqrt{299}/3$
$b = 6$	×	×	×	$4\sqrt{7}$	16	$12\sqrt{3}$	$8\sqrt{10}$	$4\sqrt{55}$	$24\sqrt{2}$
$b = 7$	×	×	×	$8\sqrt{15}/3$	$8\sqrt{51}/3$	$8\sqrt{95}/3$	$56\sqrt{3}/3$	$7\sqrt{23}$	$35\sqrt{11}/3$
$b = 8$	×	×	×	×	16	$8\sqrt{5}/3$	$16\sqrt{33}/3$	$64\sqrt{3}/3$	$16\sqrt{65}/3$
$/3b = 9$	×	×	×	×	$3\sqrt{19}$	$9\sqrt{7}$	$3\sqrt{115}$	$15\sqrt{7}$	$27\sqrt{3}$
right triangles	$s_{rt}(a,b;c) = \dfrac{ab}{2(1-c^2)}$								$s_{rt}(a,b;0.5) = \dfrac{2ab}{3}$
$s_{rt}(a,b;0.5)$	$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$	$a = 6$	$a = 7$	$a = 8$	$a = 9$
$b = 1$	2/3	4/3	2	8/3	10/3	4	14/3	16/3	6
$b = 2$	4/3	8/3	4	16/3	20/3	8	28/3	32/3	12
$b = 3$	2	4	6	8	10	12	14	16	18
$b = 4$	8/3	16/3	8	32/3	40/3	16	56/3	64/3	24
$b = 5$	10/3	20/3	10	40/3	50/3	20	70/3	80/3	30
$b = 6$	4	8	12	16	20	24	28	32	36
$b = 7$	14/3	28/3	14	56/3	70/3	28	98/3	112/3	42
$b = 8$	16/3	32/3	16	64/3	80/3	32	112/3	128/3	48
$b = 9$	6	12	18	24	30	36	42	48	54
rectangles	$s_{re}(a,b;c) = \dfrac{ab}{1-c^2}$								$s_{re}(a,b;0.5) = \dfrac{4ab}{3}$

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$s_{re}(a,b;0.5)$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$b=1$	$4/3$	$8/3$	4	$16/3$	$20/3$	8	$28/3$	$32/3$	12
$b=2$	$8/3$	$16/3$	8	$32/3$	$40/3$	16	$56/3$	$64/3$	24
$b=3$	4	8	12	16	20	24	28	32	36
$b=4$	$16/3$	$32/3$	16	$64/3$	$80/3$	32	$112/3$	$128/3$	48
$b=5$	$20/3$	$40/3$	20	$80/3$	$100/3$	40	$140/3$	$160/3$	60
$b=6$	8	16	24	32	40	48	56	64	72
$b=7$	$28/3$	$56/3$	28	$112/3$	$140/3$	56	$196/3$	$224/3$	84
$b=8$	$32/3$	$64/3$	32	$128/3$	$160/3$	64	$224/3$	$256/3$	96
$b=9$	12	24	36	48	60	72	84	96	108

Source: own modelling in Maple 2022

Table 2 Some values of the sums $s_{sq}(a;0.5), s_{rh}(a;0.5), s_{ro}(a;0.5), s_{np}(a,n;0.5)$, $s_{ci}(a;0.5)$ and $s_{el}(a,b;0.5)$

squares	$s_{sq}(a;c)=\dfrac{a^2}{1-c^2}$								$s_{sq}(a;0.5)=\dfrac{4a^2}{3}$
	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$s_{sq}(a;0.5)$	$4/3$	$16/3$	12	$64/3$	$100/3$	48	$196/3$	$256/3$	108
regular hexagons	$s_{rh}(a;c)=\dfrac{3\sqrt{3}a^2}{2(1-c^2)}$								$s_{rh}(a;0.5)=2\sqrt{3}a^2$
	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$s_{rh}(a;0.5)$	$2\sqrt{3}$	$8\sqrt{3}$	$18\sqrt{3}$	$32\sqrt{3}$	$50\sqrt{3}$	$72\sqrt{3}$	$98\sqrt{3}$	$128\sqrt{3}$	$162\sqrt{3}$
regular octagons	$s_{ro}(a;c)=\dfrac{2(1+\sqrt{2})a^2}{1-c^2}$			$[k=1+\sqrt{2}]$					$s_{ro}(a;0.5)=\dfrac{8(1+\sqrt{2})a^2}{3}$
	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$s_{et}(a;0.5)$	$8k/3$	$32k/3$	$24k$	$128k/3$	$200k/3$	$96k$	$392k/3$	$512k/3$	$216k$
***n*-sided regular polygons**	$s_{np}(a,n;c)=\dfrac{n\cot(\pi/n)\,a^2}{4(1-c^2)}$								$s_{np}(a,n;0.5)=\dfrac{na^2}{3\tan(\pi/n)}$
$s_{np}(a,n;0.5)$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$n=5$	2.294	9.716	20.646	36.704	57.349	82.583	112.404	146.814	185.811
$n=6$	3.464	13.857	31.178	55.426	86.603	124.708	169.741	221.702	280.592
$n=7$	4.845	19.381	43.606	77.524	121.131	174.429	237.416	310.094	392.464
$n=8$	6.438	25.752	57.941	103.007	160.948	231.765	315.459	412.028	521.472
$n=9$	8.243	32.971	74.185	131.880	206.062	296.729	403.881	527.518	667.640
$n=10$	10.259	41.035	92.331	164.143	256.474	369.323	502.688	656.573	830.976
$n=11$	12.487	49.950	112.388	199.801	312.188	449.551	611.890	799.203	1011.49
$n=12$	14.928	59.712	134.354	238.851	373.205	537.415	731.482	955.405	1209.18
$n=16$	26.812	107.250	241.312	428.999	670.312	965.249	1313.81	1715.99	2171.81
circles	$s_{ci}(a;c)=\dfrac{\pi a^2}{1-c^2}$								$s_{ci}(a;0.5)=\dfrac{4\pi a^2}{3}$
	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$s_{ci}(a;0.5)$	$4\pi/3$	$16\pi/3$	12π	$64\pi/3$	$100\pi/3$	48π	$196\pi/3$	$256\pi/3$	108π

ellipses	$s_{el}(a,b;c)=\frac{\pi ab}{1-c^2}$						$s_{el}(a,b;0.5)=\frac{4\pi ab}{3}$		
$s_{el}(a,b;0.5)$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$
$b=1$	$4\pi/3$	$8\pi/3$	4π	$16\pi/3$	$20\pi/3$	8π	$28\pi/3$	$32\pi/3$	12π
$b=2$	$8\pi/3$	$16\pi/3$	8π	$32\pi/3$	$40\pi/3$	16π	$56\pi/3$	$64\pi/3$	24π
$b=3$	4π	8π	12π	16π	20π	24π	28π	32π	36π
$b=4$	$16\pi/3$	$32\pi/3$	16π	$64\pi/3$	$80\pi/3$	32π	$112\pi/3$	$128\pi/3$	48π
$b=5$	$20\pi/3$	$40\pi/3$	20π	$80\pi/3$	$100\pi/3$	40π	$140\pi/3$	$160\pi/3$	60π
$b=6$	8π	16π	24π	32π	40π	48π	56π	64π	72π
$b=7$	$28\pi/3$	$56\pi/3$	28π	$112\pi/3$	$140\pi/3$	56π	$196\pi/3$	$224\pi/3$	84π
$b=8$	$32\pi/3$	$64\pi/3$	32π	$128\pi/3$	$160\pi/3$	64π	$224\pi/3$	$256\pi/3$	96π
$b=9$	12π	24π	36π	48π	60π	72π	84π	96π	108π

Source: own modelling in Maple 2022

CONCLUSIONS

We dealt with the sum of the areas enclosed by ten basic planar shapes – an equilateral tringle, an isosceles triangle, a right triangle, a square, a rectangle, a regular hexagon, a regular octagon, a regular n-sided polygon, a circle and an ellipse. We illustrated these ten individual cases with pictures created with the MetaPost programming language. Furthermore, we derived these sums for arbitrary reduction coefficient $c \in (0,1)$ and especially for the value $c = 0.5$.

For this value of the reduction coefficient, we also computed, by using the computer algebra system Maple, 435 values of these sums.

This paper thus brings a relatively extensive database of problems for counting problems from the thematic set Infinite series, which can be used in teaching at secondary schools or universities.

ACKNOWLEDGEMENTS

The work presented in this paper was supported by the Project for the Development of the Organization DZRO "Military autonomous and robotic systems" under Ministry of Defence and Armed Forces of Czech Republic.

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