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Original Paper

The sum of the series of reciprocals of the cubic polynomials with one zero and two different positive integer roots

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ABSTRACT

This contribution is a follow-up to five preceding author's papers and deals with the sum of the series of reciprocals of the cubic polynomials with one zero and two different positive integer roots. We derive the formula for the sum of these series and verify it by some examples using the basic programming language of the computer algebra system Maple 2020.

KEYWORDS: sum of the series, harmonic numbers, telescoping series, computer algebra system Maple 2020

JEL CLASSIFICATION: I30

INTRODUCTION

Let us recall some used basic terms concerning infinite series. We say that a series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

converges to a limit s if and only if the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$.
We than say that the series has a *sum* s and write

$$\sum_{k=1}^{\infty} a_k = s.$$

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions (see e.g. [9]). For example the sum of the reciprocals

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- of the square numbers (the *Basel problem*)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

is $\pi^2/6$, and equals approximately 1.644934,

- of the cubes

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

is called *Apéry's constant* $\zeta(3)$, and equals approximately 1.202057,

- of the factorials

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

is the transcendental number e $\doteq 2.718282$.

In contrast to these three convergent series, for example, the following two series of the reciprocals diverge:

- the series of the reciprocals of positive integers (the *harmonic series*)

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

- and the series of the reciprocals of all prime numbers *P*

$$\sum_{p \in P}^{\infty} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots .$$

Next, we will use *harmonic numbers*, where the nth *harmonic number* is the sum of the reciprocals of the first n positive integers:

$$H(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Basic and as well interesting information about harmonic numbers can be found in [1], [8]. The values of the harmonic numbers $H(n)$ for $n = 1,2,\dots,10$ are stated in the following Table 1.

Table 1 First ten harmonic numbers

$\boldsymbol{n}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{H(n)}$	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$

Source: own modelling in Maple 2020

Let us note that there exists the Euler's constant $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$.

In addition to the harmonic numbers, we will also use telescoping series. The *telescoping series* is any series where nearly every term cancels with a preceding or following term, so its partial sums eventually only have a fixed number of terms after cancellation. Interesting facts about telescoping series, we can found in [2].

TELESCOPING SERIES EXAMPLE

Let us consider the following example, in which we determine the sum of the telescoping series formed by reciprocals of the cubic polynomial with one zero and two different positive integer roots.

Example 1 Using the nth partial sum calculate the sum $s(0,5,8)$ of the series

$$\sum_{\substack{k=1 \\ k\neq 5,8}}^{\infty} \frac{1}{k(k-5)(k-8)} \tag{1}$$

representing the telescoping series formed by reciprocals of the cubic polynomials with the integer roots $k_1 = 0$, $k_2 = 5$, and $k_3 = 8$.

By means of the result of the Maple 2020 command

```
> convert(1/(k*(k-5)*(k-8)),parfrac);
```

$$\frac{1}{24(k-8)} + \frac{1}{40k} - \frac{1}{15(k-5)}$$

we get the partial fraction decomposition of the kth term, where $k \neq 5$ and $k \neq 8$,

$$a_k = \frac{1}{k(k-5)(k-8)}$$

in the form

$$a_k = \frac{1}{40k} - \frac{1}{15(k-5)} + \frac{1}{24(k-8)} = \frac{1}{120}\left(\frac{3}{k} - \frac{8}{k-5} + \frac{5}{k-8}\right).$$

The sum in parentheses we express as the reciprocal of the integers:

$$a_k = \frac{1}{120}\left(\frac{3}{k} - \frac{5+3}{k-5} + \frac{5}{k-8}\right) = \frac{3}{120}\left(\frac{1}{k} - \frac{1}{k-5}\right) - \frac{5}{120}\left(\frac{1}{k-5} - \frac{1}{k-8}\right),$$

i.e.

$$a_k = \frac{1}{40}\left(\frac{1}{k} - \frac{1}{k-5}\right) - \frac{1}{24}\left(\frac{1}{k-5} - \frac{1}{k-8}\right).$$

The nth partial sum $s_n(0,5,8)$ of the series (1) is

$$s_n(0,5,8) = \frac{1}{40}\sum_{\substack{k=1 \\ k\neq 5,8}}^{n}\left(\frac{1}{k} - \frac{1}{k-5}\right) - \frac{1}{24}\sum_{\substack{k=1 \\ k\neq 5,8}}^{n}\left(\frac{1}{k-5} - \frac{1}{k-8}\right) = \frac{1}{40}s_n(0,5) - \frac{1}{24}s_n(5,8),$$

where

$$s_n(0,5) = \left(\frac{1}{1} + \frac{1}{4}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{1}\right) + \left(\frac{1}{6} - \frac{1}{1}\right) + \left(\frac{1}{7} - \frac{1}{2}\right) +$$

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$$+\left(\frac{1}{9}-\frac{1}{4}\right)+\left(\frac{1}{10}-\frac{1}{5}\right)+\left(\frac{1}{11}-\frac{1}{6}\right)+\left(\frac{1}{12}-\frac{1}{7}\right)+\left(\frac{1}{13}-\frac{1}{8}\right)+\left(\frac{1}{14}-\frac{1}{9}\right)+\left(\frac{1}{15}-\frac{1}{10}\right)+\cdots$$
$$\cdots+\left(\frac{1}{n-6}-\frac{1}{n-11}\right)+\left(\frac{1}{n-5}-\frac{1}{n-10}\right)+\left(\frac{1}{n-4}-\frac{1}{n-9}\right)+$$
$$+\left(\frac{1}{n-3}-\frac{1}{n-8}\right)+\left(\frac{1}{n-2}-\frac{1}{n-7}\right)+\left(\frac{1}{n-1}-\frac{1}{n-6}\right)+\left(\frac{1}{n}-\frac{1}{n-5}\right)=$$
$$=2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+$$
$$+\left(\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)$$

and

$$s_n(5,8)=\left(-\frac{1}{4}+\frac{1}{7}\right)+\left(-\frac{1}{3}+\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{1}{5}\right)+\left(-\frac{1}{1}+\frac{1}{4}\right)+\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{1}\right)+$$
$$+\left(\frac{1}{4}-\frac{1}{1}\right)+\left(\frac{1}{5}-\frac{1}{2}\right)+\left(\frac{1}{6}-\frac{1}{3}\right)+\left(\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{8}-\frac{1}{5}\right)+\left(\frac{1}{9}-\frac{1}{6}\right)+\cdots$$
$$\cdots+\left(\frac{1}{n-12}-\frac{1}{n-9}\right)+\left(\frac{1}{n-11}-\frac{1}{n-8}\right)+\left(\frac{1}{n-10}-\frac{1}{n-7}\right)+$$
$$+\left(\frac{1}{n-9}-\frac{1}{n-6}\right)+\left(\frac{1}{n-8}-\frac{1}{n-5}\right)=$$
$$=-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+2\left(\frac{1}{1}+\frac{1}{2}\right)-$$
$$-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)-\left(\frac{1}{n-7}+\frac{1}{n-6}+\frac{1}{n-5}\right).$$

Therefore, we have

$$s_n(0,5,8)=\frac{1}{40}\left[2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\right.$$
$$\left.+\left(\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)\right]-$$
$$-\frac{1}{24}\left[-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+2\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)-\right.$$
$$\left.-\left(\frac{1}{n-7}+\frac{1}{n-6}+\frac{1}{n-5}\right)\right].$$

Since for arbitrary real c it holds

$$\lim_{n\to\infty}\frac{1}{n+c}=0$$

and because

$$s(0,5,8)=\lim_{n\to\infty}s_n(0,5,8),$$

we have

$$s(0,5,8) = \frac{1}{40}\left[2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)\right]-$$
$$-\frac{1}{24}\left[-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+2\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)\right]=$$
$$=\frac{1}{40}[2H(4)+H(7)-H(5)-H(2)-H(8)+H(3)]-$$
$$-\frac{1}{24}[-H(4)+H(7)-H(3)+2H(2)-H(3)]=$$
$$=\frac{1}{40}\left(\frac{25}{6}+\frac{363}{140}-\frac{137}{60}-\frac{3}{2}-\frac{761}{280}+\frac{11}{6}\right)-\frac{1}{24}\left(-\frac{25}{12}+\frac{363}{140}-\frac{11}{3}+3\right)=$$
$$=\frac{1}{40}\cdot\frac{251}{120}-\frac{1}{24}\cdot\left(-\frac{11}{70}\right)=\frac{251}{4800}+\frac{11}{1680}=\frac{659}{11200}\doteq 0.058839\,.$$

The sum of the series (1) we can also compute by means of Maple 2020 this way:

```
> evalf(sum(1/(k*(k-5)*(k-8)),k=1..4)+sum(1/(k*(k-5)*
(k-8)),k=6..7)+ sum(1/(k*(k-5)*(k-8)),k=9..infinity),10);
```

0.058839

THREE LEMMAS

This paper is a free follow-up to author's papers [3], [4], [5], [6], [7] dealing with the sum of the telescoping series formed by reciprocals of the cubic polynomials with some positive integer roots. Before we derive the main result of this paper, we present three following lemmas:

Lemma 1 Let $a < b$ be positive integers. Then a fraction

$$\frac{1}{k(k-a)(k-b)}$$

can be rewritten in the form

$$\frac{1}{ab(b-a)}\left(\frac{b-a}{k}-\frac{(b-a)+a}{k-a}+\frac{a}{k-b}\right).$$

This expression can also be rewritten as a difference

$$\frac{1}{ab}\left(\frac{1}{k}-\frac{1}{k-a}\right)-\frac{1}{b(b-a)}\left(\frac{1}{k-a}-\frac{1}{k-b}\right). \tag{2}$$

Proof. Can be simply made in Maple using the simplify command applied to expression (2).

Lemma 2 Let $a < b$ be positive integers. Then it holds

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\left(\frac{1}{k}-\frac{1}{k-a}\right)=H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}, \tag{3}$$

where $H(n)$ is the nth harmonic number.

Proof. The sum $s(0,a)$ of the infinite series in (3) is the limit of the sequence $\{s_n(0,a)\}_{n=1}^{\infty}$of the partial sums

$$s_n(0,a)=\sum_{k=1}^{a-1}\left(\frac{1}{k}-\frac{1}{k-a}\right)+\sum_{k=a+1}^{b-1}\left(\frac{1}{k}-\frac{1}{k-a}\right)+\sum_{k=b+1}^{n}\left(\frac{1}{k}-\frac{1}{k-a}\right).$$

In fact

$$s_n(0,a)=\left(\frac{1}{1}-\frac{1}{1-a}\right)+\left(\frac{1}{2}-\frac{1}{2-a}\right)+\cdots+\left(\frac{1}{a-2}-\frac{1}{-2}\right)+\left(\frac{1}{a-1}-\frac{1}{-1}\right)+$$
$$+\left(\frac{1}{a+1}-\frac{1}{1}\right)+\left(\frac{1}{a+2}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{b-2}-\frac{1}{b-2-a}\right)+\left(\frac{1}{b-1}-\frac{1}{b-1-a}\right)+$$
$$+\left(\frac{1}{b+1}-\frac{1}{b+1-a}\right)+\left(\frac{1}{b+2}-\frac{1}{b+2-a}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n-1-a}\right)+\left(\frac{1}{n}-\frac{1}{n-a}\right)=$$
$$=2H(a-1)+H(b-1)-H(a)-H(b-1-a)+H(n)-H(b)-H(n-a)+H(b-a)=$$
$$=[2H(a-1)-H(a)]+[H(b-1)-H(b-1-a)]-[H(b)-H(b-a)]+[H(n)-H(n-a)].$$

Since $a<b$, then

$$H(b)-H(b-a)=\frac{1}{b-a+1}+\frac{1}{b-a+2}+\cdots+\frac{1}{b-1}+\frac{1}{b}$$

and we have

$$s_n(0,a)=\left[H(a-1)-\frac{1}{a}\right]+\left[\frac{1}{b-1}+\frac{1}{b-2}\cdots+\frac{1}{b-a+1}+\frac{1}{b-a}\right]-$$
$$-\left[\frac{1}{b}+\frac{1}{b-1}+\cdots+\frac{1}{b-a+2}+\frac{1}{b-a+1}\right]+\left[\frac{1}{n-a+1}+\frac{1}{n-a+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right]=$$
$$=H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}+\frac{1}{n-a+1}+\frac{1}{n-a+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}.$$

Hence, we have

$$s(0,a)=\lim_{n\to\infty}\left[H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}+\frac{1}{n-a+1}+\frac{1}{n-a+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right]=$$
$$=H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}.$$

Lemma 3 Let $a<b$ be positive integers. Then it holds

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\left(\frac{1}{k-a}-\frac{1}{k-b}\right)=-H(a-1)+H(b-1)-\frac{2}{b-a}, \tag{4}$$

where $H(n)$ is the nth harmonic number.

Proof. The sum $s(a,b)$ of the infinite series in (4) is the limit of the sequence $\{s_n(a,b)\}_{n=1}^{\infty}$ of the partial sums

$$s_n(a,b) = \sum_{k=1}^{a-1}\left(\frac{1}{k-a}-\frac{1}{k-b}\right) + \sum_{k=a+1}^{b-1}\left(\frac{1}{k-a}-\frac{1}{k-b}\right) + \sum_{k=b+1}^{n}\left(\frac{1}{k-a}-\frac{1}{k-b}\right).$$

In fact

$$s_n(a,b) = \left(\frac{1}{1-a}-\frac{1}{1-b}\right)+\left(\frac{1}{2-a}-\frac{1}{2-b}\right)+\cdots+\left(\frac{1}{-2}-\frac{1}{a-2-b}\right)+\left(\frac{1}{-1}-\frac{1}{a-1-b}\right)+$$
$$+\left(\frac{1}{1}-\frac{1}{a+1-b}\right)+\left(\frac{1}{2}-\frac{1}{a+2-b}\right)+\cdots+\left(\frac{1}{b-2-a}-\frac{1}{-2}\right)+\left(\frac{1}{b-1-a}-\frac{1}{-1}\right)+$$
$$+\left(\frac{1}{b+1-a}-\frac{1}{1}\right)+\left(\frac{1}{b+2-a}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{n-1-a}-\frac{1}{n-1-b}\right)+\left(\frac{1}{n-a}-\frac{1}{n-b}\right)=$$
$$=-\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{a-1}\right)+\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{b-a-1}+\frac{1}{b-a+1}+\cdots+\frac{1}{b-1}\right)+$$
$$+\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{b-1-a}+\frac{1}{b+1-a}+\cdots+\frac{1}{n-a}\right)-\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n-b}\right)=$$
$$=-H(a-1)+H(b-1)-\frac{1}{b-a}+H(n-a)-\frac{1}{b-a}-H(n-b).$$

Since $a<b$, then

$$H(n-a)-H(n-b)=\frac{1}{n-a}+\frac{1}{n-a-1}+\cdots+\frac{1}{n-b+1}$$

and we have

$$s_n(a,b)=-H(a-1)+H(b-1)-\frac{2}{b-a}+\frac{1}{n-a}+\frac{1}{n-a-1}+\cdots+\frac{1}{n-b+1}.$$

Hence, we have

$$s(a,b)=\lim_{n\to\infty}\left[-H(a-1)+H(b-1)-\frac{2}{b-a}+\frac{1}{n-a}+\frac{1}{n-a-1}+\cdots+\frac{1}{n-b+1}\right]=$$
$$=-H(a-1)+H(b-1)-\frac{2}{b-a}.$$

THE MAIN RESULT

Now, let us consider the series formed by reciprocals of the cubic polynomial with one zero and two different positive integer roots $a<b$, i.e. the series

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\frac{1}{k(k-a)(k-b)}, \tag{5}$$

and let us determine its sum $s(0,a,b)$.

According to Lemma 1 we can the kth term a_k of the series (5) write in the form

$$a_k = \frac{1}{ab}\left(\frac{1}{k} - \frac{1}{k-a}\right) - \frac{1}{b(b-a)}\left(\frac{1}{k-a} - \frac{1}{k-b}\right),$$

so the sum $s(0,a,b)$ of the series (5) is

$$s(0,a,b) = \frac{1}{ab}\sum_{\substack{k=1 \\ k\neq a,b}}^{n}\left(\frac{1}{k} - \frac{1}{k-a}\right) - \frac{1}{b(b-a)}\sum_{\substack{k=1 \\ k\neq a,b}}^{n}\left(\frac{1}{k-a} - \frac{1}{k-b}\right).$$

According to the Lemmas 2 and 3 we have

$$s(0,a,b) = \frac{1}{ab}\left[H(a-1) - \frac{1}{a} - \frac{1}{b} + \frac{1}{b-a}\right] - \frac{1}{b(b-a)}\left[H(b-1) - H(a-1) - \frac{2}{b-a}\right] =$$

$$= \frac{1}{ab}\left[H(a-1) + \frac{a^2+ab-b^2}{ab(b-a)}\right] - \frac{1}{b(b-a)}\left[H(b-1) - H(a-1) - \frac{2}{b-a}\right] =$$

$$= \frac{H(a-1)}{ab} + \frac{a^2+ab-b^2}{a^2b^2(b-a)} - \frac{H(b-1)-H(a-1)}{b(b-a)} + \frac{2}{b(b-a)^2} =$$

$$= \frac{H(a-1)}{ab} - \frac{H(b-1)-H(a-1)}{b(b-a)} - \frac{a^3-2a^2b-2ab^2+b^3}{a^2b^2(b-a)^2}$$

$$= \frac{H(a-1)}{a(b-a)} - \frac{H(b-1)}{b(b-a)} - \frac{a^3-2a^2b-2ab^2+b^3}{a^2b^2(b-a)^2}.$$

We have derived the following statement:

Theorem 1 The series

$$\sum_{\substack{k=1 \\ k\neq a,b}}^{\infty}\frac{1}{k(k-a)(k-b)},$$

where $a < b$ be positive integers, has the sum

$$s(0,a,b) = \frac{H(a-1)}{a(b-a)} - \frac{H(b-1)}{b(b-a)} - \frac{a^3-2a^2b-2ab^2+b^3}{a^2b^2(b-a)^2}, \tag{6}$$

where $H(n)$ is the nth harmonic number.

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(0,a,b)$ for $a = 1,2,\dots,10$ and for $b = a+1, a+2, \dots, a+10$. We use on the one hand an approximate direct evaluation of the sum

$$s(0,a,b,t) = \sum_{\substack{k=1 \\ k\neq a,b}}^{t}\frac{1}{k(k-a)(k-b)},$$

where $t = 10^7$, using the basic programming language of the computer algebra system Maple 2020, and on the other hand the formula (6) for evaluation the sum $s(0, a, b)$. We compare one hundred pairs of these ways obtained sums $s(0, a, b, t)$ and $s(0, a, b)$ to verify the formula (6). We use the following simple procedure `ts0abpos` and the following double repetition statement:

```
> ts0abpos:=proc(a,b,t)
      local k,s0ab,s0abt,s0abt1,s0abt2,s0abt3;
      s0abt1:=0; s0abt2:=0; s0abt3:=0;
      s0ab:=harmonic(a-1)/(a*(b-a))-harmonic(b-1)/(b*(b-a))
           -(a*a*a-2*a*a*b-2*a*b*b+b*b*b)/(a*a*b*b*(b-a)*(b-a));
      print("s(0",a,b,")=",evalf[12](s0ab));
      for k from 1 to a-1 do
          s0abt1:=s0abt1+1/(k*(k-a)*(k-b)); end do;
      for k from a+1 to b-1 do
          s0abt2:=s0abt2+1/(k*(k-a)*(k-b)); end do;
      for k from b+1 to t do
          s0abt3:=s0abt3+1/(k*(k-a)*(k-b)); end do;
      s0abt:=s0abt1+s0abt2+s0abt3;
      print("s(0,a,b,t)=",evalf[12](s0abt));
      print("diff=",evalf[12](abs(s0abt-s0ab)));
  end proc:
> for i from 1 to 10 do
       for j from i+1 to i+10 do ts0abpos(i,j,10000000); end do;
   end do;
```

The approximate values of the sums $s(0, a, b)$ rounded to 10 decimals obtained by this procedure are written into the following Table 2.

Table 2 Some approximate values of the sums $s(0, a, b)$ obtained by means of the formula (6)

$a \setminus b$	$b = a + 1$	$b = a + 2$	$b = a + 3$	$b = a + 4$	$b = a + 5$
$a = 1$	0.2500000000	−0.3611111111	−0.3263888889	−0.2691666667	−0.2238888889
$a = 2$	0.6944444444	0.1145833333	0.0355555556	0.0159722222	0.0097959184
$a = 3$	0.5763888889	0.1394444444	0.0675925926	0.0445861678	0.0344146825
$a = 4$	0.4691666667	0.1256944444	0.0657312925	0.0452752976	0.0356834215
$a = 5$	0.3905555556	0.1092517007	0.0588392857	0.0412014991	0.0327539683
$a = 6$	0.3327097506	0.0950644841	0.0519106408	0.0366369048	0.0292575102
$a = 7$	0.2889668367	0.0834687579	0.0458824641	0.0325070042	0.0260235862
$a = 8$	0.2549548060	0.0740426587	0.0408154871	0.0289633688	0.0232150809
$a = 9$	0.2278527337	0.0663215853	0.0365804206	0.0259639103	0.0208192028
$a = 10$	0.2058005378	0.0599257456	0.0330266101	0.0234258995	0.0187807563
$a \setminus b$	$b = a + 6$	$b = a + 7$	$b = a + 8$	$b = a + 9$	$b = a + 10$
$a = 1$	−0.1898526077	−0.1639668367	−0.1438436949	−0.1278527337	−0.1148914469
$a = 2$	0.0076140873	0.0068090199	0.0065128968	0.0064056875	0.0063621332
$a = 3$	0.0288874192	0.0254138322	0.0229892267	0.0211636871	0.0197123176
$a = 4$	0.0302116402	0.0266470362	0.0240984999	0.0221518629	0.0205925820
$a = 5$	0.0278576894	0.0246329795	0.0223113372	0.0223113372	0.0191006956
$a = 6$	0.0249554072	0.0221121369	0.0200611640	0.0184863696	0.0172211382
$a = 7$	0.0222384199	0.0197362445	0.0179319919	0.0165473594	0.0154353700
$a = 8$	0.0198616446	0.0176481026	0.0160545726	0.0148334195	0.0138537867

$a = 9$	0.0178236167	0.0158509962	0.0144342593	0.0133507928	0.0124829743
$a = 10$	0.0160828494	0.0143113853	0.0130426798	0.0120747685	0.0113009843

Source: own modelling in Maple 2020

Computation of 100 pairs of the sums $s(0,a,b)$ and $s(0,a,b,10^7)$ took over 34 minutes. The absolute errors, i.e. the differences $|s(0,a,b) - s(0,a,b,10^7)|$, are all only about $5 \cdot 10^{-15}$.

CONCLUSIONS

We dealt with the sum of the telescoping series formed by reciprocals of the cubic polynomials with one zero and two different positive integer roots $0 < a < b$. We derived that the sum $s(0,a,b)$ of this series is given by the formula

$$s(0,a,b) = \frac{H(a-1)}{a(b-a)} - \frac{H(b-1)}{b(b-a)} - \frac{a^3 - 2a^2b - 2ab^2 + b^3}{a^2b^2(b-a)^2},$$

where $H(n)$ is the nth harmonic number. This series so belong to special types of infinite series, such as geometric series, which sums are given analytically by means of a simple formula. This series so belong to special types of infinite series, such as geometric series, which sums are given analytically by means of a simple formula.

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