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Original Paper

Evaluation of specific integrals by differentiation – part 3

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ABSTRACT

The technique of integration (anti-differentiation) represents one of the most important techniques in calculus. While its counterpart, differentiation, is a routine and relatively simple procedure, integration, in general, is a much more involving task. The inverse relationship between differentiation and anti-differentiation (evaluation of indefinite integrals) in some particular cases reveals the possibility to derive the form of the antiderivative and evaluate this antiderivative by differentiation and subsequent comparison of coefficients. This paper is a sequel to previous author's papers and deals with some other types of elementary functions whose indefinite integrals can be, at least partly, evaluated by differentiation and comparison of coefficients.

KEYWORDS: higher mathematics, differentiation, integration, undetermined coefficients

JEL CLASSIFICATION: I 20, C20

INTRODUCTION

In [2], we investigated integrals containing polynomials and various rational powers of a linear function. Namely, we discussed the antiderivatives of the following functions: $\frac{P_n(x)}{\sqrt{ax+b}}$, $\frac{P_m(x)}{\sqrt[n]{ax+b}}$, $\frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}$, where $P_n(x)$, $P_m(x)$, $P_r(x)$ denote polynomials of *n*-th, *m*-th and *r*-th degree, respectively. Let's recall the results from [2]. We showed that the indefinite integrals of these functions can be expressed as follows:

$$\int \frac{P_n(x)}{\sqrt{ax+b}}\,dx = Q_n(x)\sqrt{ax+b}$$

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$$\int \frac{P_m(x)}{\sqrt[n]{ax+b}}\,dx = Q_m(x)\sqrt[n]{(ax+b)^{n-1}}$$

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}\,dx = Q_r(x)\sqrt[n]{(ax+b)^{n-m}}$$

For example

$$\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x = (Ax^2+Bx+C)\sqrt[4]{(x-1)^5}$$

Upon differentiation and comparison of coefficients we obtained the antiderivative without "classical" integration.

$$\int (x^2+1)\sqrt[4]{x-1}\,dx = \left(\frac{4}{13}x^2+\frac{32}{117}x+\frac{596}{585}\right)\sqrt[4]{(x-1)^5}+const$$

In this paper we investigate integrals containing polynomials and various rational powers of polynomials.

RESULTS AND DISCUSSION

Like in [1] and [2], we denote polynomials of degree *n*, *m*, *r* as $P_n(x), Q_m(x), \alpha_r(x)$ etc., respectively and their *k*-th derivatives as $P_{n-k}(x), Q_{m-k}(x), \alpha_{r-k}(x)$ etc., respectively, further *a*, *b*, *c* etc. are given (real) constants and *A*, *B*, *C*, α, β etc. are unknown coefficients. All the investigated integrals are considered on intervals where they are defined and in all cases and illustrative examples we set the integration constant equal to zero.

Let us consider the simplest case first.

1. $\displaystyle\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx.$

Since the square root of a quadratic polynomial is transformed by differentiation into its reciprocal multiplied by a linear term, we will consider the function $Q_n(x)\sqrt{ax^2+bx+c}$ and its derivative. In all discussions in this paper, except for the illustrative examples, we consider polynomials in general, hence we deliberately neglect the constant multiples of polynomials that arise during differentiation.

$$\left[Q_n(x)\sqrt{ax^2+bx+c}\right]' = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \frac{Q_n(x)(2ax+b)}{\sqrt{ax^2+bx+c}}$$

$$Q_n(x)\sqrt{ax^2+bx+c} = \int Q_{n-1}(x)\sqrt{ax^2+bx+c} + \frac{Q_n(x)(2ax+b)}{\sqrt{ax^2+bx+c}}\,dx$$

$$Q_n(x)\sqrt{ax^2+bx+c}=\int \frac{Q_{n-1}(x)\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}+\frac{Q_n(x)(2ax+b)}{\sqrt{ax^2+bx+c}}\,dx$$

Now let $P_{n+1}(x)=Q_{n-1}(x)\left(ax^2+bx+c\right)+Q_n(x)(2ax+b)$, and we can write

$$\int \frac{P_{n+1}(x)}{\sqrt{ax^2+bx+c}}\,dx=Q_n(x)\sqrt{ax^2+bx+c}$$

But the polynomials $P_{n+1}(x), Q_n(x)$ differ in degree and $Q_n(x)$ contains fewer (unknown) coefficients than $P_{n+1}(x)$, so it is necessary to make up for the missing coefficient. This coefficient (in order to be "preserved" by differentiation) assumes the form

$\int \frac{\alpha}{\sqrt{ax^2+bx+c}}\,dx,$ hence for $n \geq 0$ we get

$$\int \frac{P_{n+1}(x)}{\sqrt{ax^2+bx+c}}\,dx=Q_n(x)\sqrt{ax^2+bx+c}+\alpha\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

Example 1: Evaluate $\int \frac{2x+4}{\sqrt{x^2+1}}\,dx$.

Solution:

$$\int \frac{2x+4}{\sqrt{x^2+1}}\,dx=A\sqrt{x^2+1}+\alpha\int \frac{1}{\sqrt{x^2+1}}\,dx$$

Now we take the derivatives of both sides

$$\frac{2x+4}{\sqrt{x^2+1}}=\frac{2Ax}{2\sqrt{x^2+1}}+\frac{\alpha}{\sqrt{x^2+1}}$$

$$2x+4=Ax+\alpha$$

We see that $A=2, \alpha=4$, hence

$$\int \frac{2x+4}{\sqrt{x^2+1}}\,dx=2\sqrt{x^2+1}+4\int \frac{1}{\sqrt{x^2+1}}\,dx\ .$$

Of course, the right hand side integral should further be evaluated. We do so only in this particular case. Since it is tabulated, we get

$$\int \frac{2x+4}{\sqrt{x^2+1}}\,dx=2\sqrt{x^2+1}+4\ln\left|x+\sqrt{x^2+1}\right|$$

Example 2: Evaluate $\int \frac{2x^3+3x+1}{\sqrt{x^2+4x-5}}\,dx$.

Solution:

$$\int \frac{2x^3+3x+1}{\sqrt{x^2+4x-5}}\,dx = \left(Ax^2+Bx+C\right)\sqrt{x^2+4x-5}+\alpha\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$$

$$\frac{2x^3+3x+1}{\sqrt{x^2+4x-5}} = \left(2Ax+B\right)\sqrt{x^2+4x-5}+\frac{\left(Ax^2+Bx+C\right)\left(2x+4\right)}{2\sqrt{x^2+4x-5}}+\frac{\alpha}{\sqrt{x^2+4x-5}}$$

$$2x^3+3x+1=\left(2Ax+B\right)\left(x^2+4x-5\right)+\left(Ax^2+Bx+C\right)\left(x+2\right)+\alpha$$

Now we distribute and compare the coefficients

$$2x^3+3x+1=2Ax^3+Bx^2+8Ax^2+4Bx-10Ax-5B+Ax^3+Bx^2+Cx+2Ax^2+2Bx+2C+\alpha$$

$$A=\frac{2}{3}, B=-\frac{10}{3}, C=\frac{89}{3}, D=-75$$

$$\int \frac{2x^3+3x+1}{\sqrt{x^2+4x-5}}\,dx = \left(\frac{2}{3}x^2-\frac{10}{3}x+\frac{89}{3}\right)\sqrt{x^2+4x-5}-75\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$$

Now we generalize this case to the *m*-th root of the polynomial in the denominator.

2. $\int \frac{P_n(x)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$ and $\int \frac{P_n(x)}{\sqrt[m]{ax^2+bx+c}}\,dx$.

We derive the form of the antiderivative for the first integral.

$$\left[Q_n(x)\sqrt[m]{ax^2+bx+c}\right]' = Q_{n-1}(x)\sqrt[m]{ax^2+bx+c}+\frac{Q_n(x)(2ax+b)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}$$

$$\left[Q_n(x)\sqrt[m]{ax^2+bx+c}\right] = \int Q_{n-1}(x)\sqrt[m]{ax^2+bx+c}+\frac{Q_n(x)(2ax+b)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$$

$$\left[Q_n(x)\sqrt[m]{ax^2+bx+c}\right] = \int \frac{Q_{n-1}(x)\left(ax^2+bx+c\right)+Q_n(x)(2ax+b)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$$

Let again $P_{n+1}(x)=Q_{n-1}(x)\left(ax^2+bx+c\right)+Q_n(x)(2ax+b)$

$\int \frac{P_{n+1}(x)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx = Q_n(x)\sqrt[m]{ax^2+bx+c}$ and we again fix the missing coefficient, so

$$\int \frac{P_{n+1}(x)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx = Q_n(x)\sqrt[m]{ax^2+bx+c} + \alpha\int \frac{1}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$$

Analogously for $\int \frac{P_n(x)}{\sqrt[m]{ax^2+bx+c}}\,dx$ we would get

$$\int \frac{P_{n+1}(x)}{\sqrt[m]{ax^2+bx+c}}\,dx = Q_n(x)\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}} + \alpha\int \frac{1}{\sqrt[m]{ax^2+bx+c}}\,dx.$$

Both these integrals are valid for $n \geq 0$ and $m \neq 0, 1$.

Example 3: Evaluate $\int \frac{x^2-3x}{\sqrt[5]{x^2+x-1}}\,dx$.

Solution:

$$\int \frac{x^2-3x}{\sqrt[5]{x^2+x-1}}\,dx = (Ax+B)\sqrt[5]{\left(x^2+x-1\right)^4} + \alpha\int \frac{1}{\sqrt[5]{x^2+x-1}}\,dx,$$

and differentiate

$$\frac{x^2-3x}{\sqrt[5]{x^2+x-1}} = A\sqrt[5]{\left(x^2+x-1\right)^4} + \frac{4}{5}\frac{(Ax+B)(2x+1)}{\sqrt[5]{x^2+x-1}} + \frac{\alpha}{\sqrt[5]{x^2+x-1}},$$

then a little algebra

$$\frac{x^2-3x}{\sqrt[5]{x^2+x-1}} = \frac{5A\left(x^2+x-1\right)+4(Ax+B)(2x+1)+5\alpha}{5\sqrt[5]{x^2+x-1}}$$

and finally we distribute and compare the coefficients

$$5x^2-15x = 5A\left(x^2+x-1\right)+4(Ax+B)(2x+1)+5\alpha$$

$$5x^2-15x = 5Ax^2+5Ax-5A+8Ax^2+8Bx+4Ax+4B+5\alpha$$

Upon evaluation of the coefficients we have

$$\int \frac{x^2-3x}{\sqrt[5]{x^2+x-1}}\,dx = \left(\frac{5}{13}x-\frac{30}{13}\right)\sqrt[5]{\left(x^2+x-1\right)^4} + \frac{29}{13}\int \frac{1}{\sqrt[5]{x^2+x-1}}\,dx$$

Next we generalize to the case of the *m*-th root of the *r*-th degree polynomial in the denominator.

3. $\int \frac{P_n(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}\,dx$ and $\int \frac{P_n(x)}{\sqrt[m]{S_r(x)}}\,dx$.

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Again we derive the form of the antiderivative for the first integral.

$$\left[Q_n(x)\sqrt[m]{S_r(x)}\right]' = Q_{n-1}(x)\sqrt[m]{S_r(x)} + \frac{Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{(S_r(x))^{m-1}}}$$

$$Q_n(x)\sqrt[m]{S_r(x)} = \int Q_{n-1}(x)\sqrt[m]{S_r(x)} + \frac{Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{(S_r(x))^{m-1}}}\,dx$$

$$Q_n(x)\sqrt[m]{S_r(x)} = \int \frac{Q_{n-1}(x)\,S_r(x) + Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{(S_r(x))^{m-1}}}\,dx$$

Let $P_{n+r-1}(x) = Q_{n-1}(x)\,S_r(x) + Q_n(x)S_{r-1}(x)$, then

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{(S_r(x))^{m-1}}}\,dx = Q_n(x)\sqrt[m]{S_r(x)}$$

Upon completion of the missing coefficients we get

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{(S_r(x))^{m-1}}}\,dx = Q_n(x)\sqrt[m]{S_r(x)} + \int \frac{\alpha_{r-2}(x)}{\sqrt[m]{(S_r(x))^{m-1}}}\,dx$$

And for $\int \frac{P_n(x)}{\sqrt[m]{S_r(x)}}\,dx$ we have

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{S_r(x)}}\,dx = Q_n(x)\sqrt[m]{(S_r(x))^{m-1}} + \int \frac{\alpha_{r-2}(x)}{\sqrt[m]{S_r(x)}}\,dx$$

Note that $r \geq 2$, $n \geq 0$ and $m \neq 0, 1$.

Now let us replace *m* by –*m*, then

$$\int P_{n+r-1}(x)\sqrt[m]{S_r(x)}\,dx = Q_n(x)\sqrt[m]{(S_r(x))^{m+1}} + \int \alpha_{r-2}(x)\sqrt[m]{S_r(x)}\,dx$$

Example 4: Evaluate $\int \frac{x^2+4x+1}{\sqrt[4]{x^3+x}}\,dx$.

Solution:

$$\int \frac{x^2+4x+1}{\sqrt[4]{x^3+x}}\,dx = A\sqrt[4]{(x^3+x)^3} + \int \frac{\alpha\, x+\beta}{\sqrt[4]{x^3+x}}\,dx$$

We differentiate again

$$\frac{x^2+4x+1}{\sqrt[4]{x^3+x}} = A\frac{3}{4}\frac{3x^2+1}{\sqrt[4]{x^3+x}} + \frac{\alpha\, x+\beta}{\sqrt[4]{x^3+x}}$$

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from which

$$x^2+4x+1=\frac{9}{4}Ax^2+\frac{3}{4}A+\alpha\,x+\beta$$

$$\int\frac{x^2+4x+1}{\sqrt[4]{x^3+x}}\,dx=\frac{4}{9}\sqrt[4]{\left(x^3+x\right)^3}+\int\frac{4\,x+\frac{2}{3}}{\sqrt[4]{x^3+x}}\,dx\,.$$

And finally we consider the most general case, i.e. the general rational power of the r-th degree polynomial in the denominator.

4. $\int\frac{P_n(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-k}}}\,dx$ and $\int\frac{P_n(x)}{\sqrt[m]{\left(S_r(x)\right)^{k}}}\,dx$.

Note again that $r\geq 2$, $n\geq 0$ and $m\neq 0$.

As in the previous cases we differentiate the function

$$\left[Q_n(x)\sqrt[m]{\left(S_r(x)\right)^k}\right]'=Q_{n-1}(x)\sqrt[m]{\left(S_r(x)\right)^k}+\frac{Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-k}}}$$

$$Q_n(x)\sqrt[m]{\left(S_r(x)\right)^k}=\int\frac{Q_{n-1}(x)\,S_r(x)+Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-k}}}\,dx$$

Let $P_{n+r-1}(x)=Q_{n-1}(x)\,S_r(x)+Q_n(x)S_{r-1}(x)$, then

$\int\frac{P_{n+r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-k}}}\,dx=Q_n(x)\sqrt[m]{\left(S_r(x)\right)^k}$ and with the missing coefficients

$$\int\frac{P_{n+r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-k}}}\,dx=Q_n(x)\sqrt[m]{\left(S_r(x)\right)^k}+\int\frac{\alpha_{r-2}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-k}}}\,dx$$

It is easy to show that $\int\frac{P_n(x)}{\sqrt[m]{\left(S_r(x)\right)^{k}}}\,dx$ assumes the form

$$\int\frac{P_{n+r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{k}}}\,dx=Q_n(x)\sqrt[m]{\left(S_r(x)\right)^{m-k}}+\int\frac{\alpha_{r-2}(x)}{\sqrt[m]{\left(S_r(x)\right)^{k}}}\,dx$$

Again let us replace m by $-m$, then

$$\int P_{n+r-1}(x)\sqrt[m]{\left(S_r(x)\right)^k}\,dx=Q_n(x)\sqrt[m]{\left(S_r(x)\right)^{m+k}}+\int\alpha_{r-2}(x)\sqrt[m]{\left(S_r(x)\right)^k}\,dx$$

Example 5: Evaluate $\int \frac{x^3+2x^2-4x+5}{\sqrt[3]{(x^2-4)^5}}\,dx$.

Solution:

$$\int \frac{x^3+2x^2-4x+5}{\sqrt[3]{(x^2-4)^5}}\,dx = (Ax^2+Bx+C)\sqrt[3]{(x^2-4)^{-2}} + \int \frac{\alpha}{\sqrt[3]{(x^2-4)^5}}\,dx$$

$$\frac{x^3+2x^2-4x+5}{\sqrt[3]{(x^2-4)^5}} = \frac{(2Ax+B)}{\sqrt[3]{(x^2-4)^2}} - \frac{2}{3}\frac{(Ax^2+Bx+C)2x}{\sqrt[3]{(x^2-4)^5}} + \frac{\alpha}{\sqrt[3]{(x^2-4)^5}}$$

$$x^3+2x^2-4x+5 = (2Ax+B)(x^2-4) - \frac{2}{3}(Ax^2+Bx+C)2x+\alpha$$

$$x^3+2x^2-4x+5 = \frac{2}{3}Ax^3 - \frac{1}{3}Bx^2 - 8Ax - \frac{4}{3}Cx - 4B + \alpha$$

$$\int \frac{x^3+2x^2-4x+5}{\sqrt[3]{(x^2-4)^5}}\,dx = \left(\frac{3}{2}x^2-6x-6\right)\sqrt[3]{(x^2-4)^{-2}} - \int \frac{19}{\sqrt[3]{(x^2-4)^5}}\,dx$$

CONCLUSIONS

In the paper we investigated integrals containing polynomials and various rational powers of polynomials. The algebraic limitations of the method developed in the paper do not allow us to evaluate the "undone" integrals on the right hand side. These integrals are, in general, non-elementary and their evaluation requires, except for special cases, more sophisticated methods which are out of the scope of this paper.

However, the method presented in the paper simplifies the (reduces the degree) polynomial in the numerator and reveals the structure of the required antiderivative.

The use of the presented method is left to the reader in every particular case.

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