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Original Paper

Evaluation of specific integrals by differentiation – part 2

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ABSTRACT

One of the most important computational techniques in higher mathematics is differentiation and its counterpart, integration (anti-differentiation). While differentiation is a routine and relatively simple procedure, integration, in general, is a much more involving task. Close (inverse) relationship between differentiation and anti-differentiation (evaluation of indefinite integrals) in some cases reveals the possibility to derive the form of the antiderivative and evaluate this antiderivative by differentiation and subsequent comparison of coefficients. This paper is a sequel to [4] and deals with some other types of elementary functions whose integrals can be evaluated by differentiation.

KEYWORDS: higher mathematics, differentiation, integration, undetermined coefficients

JEL CLASSIFICATION: I20, C20

INTRODUCTION

Integration is a widely used technique in calculus. Various standard methods of evaluation of elementary indefinite integrals, e. g. tabular, substitution, integration by parts or partial fraction decomposition can be found in [1], [2], [5]. In [4], motivated by [3], we investigated a family of simple elementary functions whose antiderivatives can be found by differentiation and comparison of coefficients. Namely, we discussed the antiderivatives of the following functions:

$P_n(x)\,\mathrm{e}^{ax}, P_n(x)\sin ax, P_n(x)\cos ax, \mathrm{e}^{ax}\sin bx, \mathrm{e}^{ax}\cos bx,$

where $P_n(x)$ denotes a polynomial of n–th degree.

Let's recall the most general case from [4].

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We showed that

$$\int P_n(x)\, e^{ax} \sin bx\, dx = Q_n(x)\, e^{ax} \sin bx + R_n(x)\, e^{ax} \cos bx$$

$$\int P_m(x)\, e^{ax} \cos bx\, dx = Q_m(x)\, e^{ax} \sin bx + R_m(x)\, e^{ax} \cos bx$$

By setting $P_n(x) = P_0$ we have

$$\int P_0\, e^{ax} \sin bx\, dx = Q_0\, e^{ax} \sin bx + R_0\, e^{ax} \cos bx$$

where P_0, Q_0, R_0 are constants. In like manner we can get all considered particular cases.

In this paper we try to extend the family of functions where this method is applicable and focus on integrals containing polynomials and various radicals of a linear function.

RESULTS AND DISCUSSION

Hereinafter, we denote polynomials of degree *n, m* as $P_n(x), Q_m(x)$ etc., respectively and their *k*-th derivatives as $P_{n-k}(x), Q_{m-k}(x)$ etc., respectively, further *a, b, c* etc. are given (real) constants and *A, B, C* etc. are unknown coefficients. All the investigated integrals are considered on intervals where they are defined.

Let us consider the simplest case first.

1. $\displaystyle\int \frac{P_n(x)}{\sqrt{ax+b}}\, dx.$

Since the square root of a linear term is transformed by differentiation into its reciprocal, we will consider the function $Q_n(x)\sqrt{ax+b}$ and its derivative. In all discussions in this paper, except for the illustrative examples, we consider polynomials in general (qualitative) form, hence we deliberately neglect the constant multiples of polynomials and the factor *a*.

$$\left[Q_n(x)\sqrt{ax+b}\right]' = Q_{n-1}(x)\sqrt{ax+b} + \frac{Q_n(x)}{\sqrt{ax+b}}$$

$$Q_n(x)\sqrt{ax+b} = \int Q_{n-1}(x)\sqrt{ax+b} + \frac{Q_n(x)}{\sqrt{ax+b}}\, dx$$

$$Q_n(x)\sqrt{ax+b} = \int \frac{Q_{n-1}(x)(ax+b)}{\sqrt{ax+b}} + \frac{Q_n(x)}{\sqrt{ax+b}}\, dx$$

$$\int \frac{P_n(x)}{\sqrt{ax+b}}\, dx = Q_n(x)\sqrt{ax+b} \qquad \text{where} \qquad P_n(x) = Q_{n-1}(x)(ax+b) + Q_n(x)$$

Example 1: Evaluate $\displaystyle\int \frac{3x^2 - 2x + 4}{\sqrt{x+3}}\, dx$.

Solution:

$$\int \frac{3x^2-2x+4}{\sqrt{x+3}}\,\mathrm{d}x = \left(Ax^2+Bx+C\right)\sqrt{x+3}$$

Now we take the derivatives of both sides

$$\frac{3x^2-2x+4}{\sqrt{x+3}} = (2Ax+B)\sqrt{x+3}+\frac{Ax^2+Bx+C}{2\sqrt{x+3}}$$

$$3x^2-2x+4 = (2Ax+B)(x+3)+\frac{Ax^2+Bx+C}{2} \qquad \text{from which}$$

$$2\left(3x^2-2x+4\right) = 2\left(2Ax^2+Bx+6Ax+3B\right)+Ax^2+Bx+C$$

We see that $A=\frac{6}{5}, 12A+3B=-4, 6B+C=8 \;\Rightarrow\; B=-\frac{92}{15}, C=\frac{224}{5}$, hence

$$\int \frac{3x^2-2x+4}{\sqrt{x+3}}\,\mathrm{d}x = \left(\frac{6}{5}x^2-\frac{92}{15}x+\frac{224}{5}\right)\sqrt{x+3}+const$$

Specifically, $\int \frac{P_0}{\sqrt{x+3}}\,\mathrm{d}x = A\sqrt{x+3}$

For example, $\int \frac{4}{\sqrt{x+3}}\,\mathrm{d}x = A\sqrt{x+3}$

$$\frac{4}{\sqrt{x+3}} = \frac{A}{2\sqrt{x+3}} \quad \text{and} \quad A=8 \;\Rightarrow\; \int \frac{4}{\sqrt{x+3}}\,\mathrm{d}x = 8\sqrt{x+3}+const$$

2. $\int \frac{P_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}\,\mathrm{d}x$ and $\int \frac{P_m(x)}{\sqrt[n]{ax+b}}\,\mathrm{d}x$

These cases represent slight generalizations of the previous case. We derive the form of the antiderivative for the first integral. Let us consider again

$$\left[Q_m(x)\sqrt[n]{ax+b}\right]' = Q_{m-1}(x)\sqrt[n]{ax+b}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}$$

$$\left[Q_m(x)\sqrt[n]{ax+b}\right]' = \frac{Q_{m-1}(x)\sqrt[n]{ax+b}\,\sqrt[n]{(ax+b)^{n-1}}}{\sqrt[n]{(ax+b)^{n-1}}}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}$$

$$\left[Q_m(x)\sqrt[n]{ax+b}\right]' = \frac{Q_{m-1}(x)(ax+b)}{\sqrt[n]{(ax+b)^{n-1}}}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}$$

$$Q_m(x)\sqrt[n]{ax+b} = \int \frac{Q_{m-1}(x)(ax+b)}{\sqrt[n]{(ax+b)^{n-1}}}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}\,\mathrm{d}x$$

$$\int \frac{P_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}\,\mathrm{d}x = Q_m(x)\sqrt[n]{ax+b} \quad \text{where} \quad P_m(x) = Q_{m-1}(x)(ax+b) + Q_m(x).$$

In like manner we would derive the following equality

$$\int \frac{P_m(x)}{\sqrt[n]{ax+b}}\,\mathrm{d}x = Q_m(x)\sqrt[n]{(ax+b)^{n-1}}$$

Example 2: Evaluate $\int \frac{x^2+3}{\sqrt[4]{x+1}}\,\mathrm{d}x$.

Solution:

$$\int \frac{x^2+3}{\sqrt[4]{x+1}}\,\mathrm{d}x = \left(Ax^2+Bx+C\right)\sqrt[4]{(x+1)^3}$$

$$\frac{x^2+3}{\sqrt[4]{x+1}} = (2Ax+B)\sqrt[4]{(x+1)^3} + \frac{3}{4}\frac{\left(Ax^2+Bx+C\right)}{\sqrt[4]{x+1}}$$

and by doing little algebra we get

$$4\left(x^2+3\right) = 4(2Ax+B)(x+1) + 3\left(Ax^2+Bx+C\right)$$

$$4x^2+12 = 11Ax^2 + 8Ax + 7Bx + 4B + 3C$$

Now we compare the coefficients and obtain $A = \frac{4}{11}, B = -\frac{32}{77}, C = \frac{1052}{231}$.

Hence the solution is

$$\int \frac{x^2+3}{\sqrt[4]{x+1}}\,\mathrm{d}x = \left(\frac{4}{11}x^2 - \frac{32}{77}x + \frac{1052}{231}\right)\sqrt[4]{(x+1)^3} + const$$

Next, we consider the most general case when the radicand is a linear function.

3. $\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}\,\mathrm{d}x$ and $\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{m}}}\,\mathrm{d}x$.

As in the previous case we derive the form of the antiderivative for the first integral.

Let us consider again

$$\left[Q_r(x)\sqrt[n]{(ax+b)^m}\right]' = Q_{r-1}(x)\sqrt[n]{(ax+b)^m} + \frac{Q_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}$$

$$\left[Q_r(x)\sqrt[n]{(ax+b)^m}\right]' = \frac{Q_{r-1}(x)\sqrt[n]{(ax+b)^m}\,\sqrt[n]{(ax+b)^{n-m}}}{\sqrt[n]{(ax+b)^{n-m}}} + \frac{Q_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}$$

$$\left[Q_r(x)\sqrt[n]{(ax+b)^m}\right]' = \frac{Q_{r-1}(x)(ax+b)}{\sqrt[n]{(ax+b)^{n-m}}} + \frac{Q_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}$$

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}\,\mathrm{d}x = Q_r(x)\sqrt[n]{(ax+b)^m} \quad \text{where} \quad P_r(x) = Q_{r-1}(x)(ax+b) + Q_r(x)$$

In the same way we would obtain

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}\,\mathrm{d}x = Q_r(x)\sqrt[n]{(ax+b)^{n-m}}$$

Due to the nature of differentiation and the technique being used here, there is a certain kind of symmetry between these two integrals.

Of course, the last formula can also be used to evaluate products of considered functions (depending on the sign of m). Let us replace m by $-m$, then

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{-m}}}\,\mathrm{d}x = \int P_r(x)\sqrt[n]{(ax+b)^m}\,dx = Q_r(x)\sqrt[n]{(ax+b)^{n+m}}$$

Example 3: Evaluate $\int \frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}}\,\mathrm{d}x$.

Solution:

$$\int \frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}}\,\mathrm{d}x = \frac{Ax^3 + Bx^2 + Cx + D}{\sqrt[3]{(2x+1)^2}}$$

$$\frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}} = \frac{3\left(3Ax^2 + 2Bx + C\right)\sqrt[3]{(2x+1)^2} - 4\left(Ax^3 + Bx^2 + Cx + D\right)\sqrt[3]{(2x-1)^{-1}}}{3\sqrt[3]{(2x+1)^4}} \quad ,$$

and after clearing the radicals it is the same as

$$3\left(4x^3 - x + 2\right) = 3\left(3Ax^2 + 2Bx + C\right)(2x+1) - 4\left(Ax^3 + Bx^2 + Cx + D\right)$$

$$12x^3 - 3x + 6 = 14Ax^3 + (8B + 9A)x^2 + (6B + 2C)x + 3C - 4D$$

$$\Rightarrow A = \frac{12}{14},\ 8B + 9A = 0,\ 6B + 2C = -3,\ 3C - 4D = 6$$

The solution of this system is $A = \frac{6}{7},\ B = -\frac{27}{28},\ C = \frac{39}{28},\ D = -\frac{51}{112}$

and

$$\int \frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}}\,\mathrm{d}x = \frac{\frac{6}{7}x^3 - \frac{27}{28}x^2 + \frac{39}{28}x - \frac{51}{112}}{\sqrt[3]{(2x+1)^2}} = \frac{1}{112}\left(\frac{96x^3 - 108x^2 + 156x - 51}{\sqrt[3]{(2x+1)^2}}\right) + const$$

Example 4: Evaluate $\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x$.

Solution:

$$\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x = (Ax^2 + Bx + C)\sqrt[4]{(x-1)^5}$$

$$(x^2+1)\sqrt[4]{x-1} = (2Ax+B)\sqrt[4]{(x-1)^5} + \frac{5}{4}(Ax^2+Bx+C)\sqrt[4]{x-1}$$

$$4(x^2+1) = 4(2Ax+B)(x-1) + 5(Ax^2+Bx+C)$$

$$4x^2 + 4 = 13Ax^2 + (-8A+9B)x - 4B + 5C$$

$$A = \frac{4}{13}, -8A+9B = 0, -4B+5C = 4 \Rightarrow B = \frac{32}{117}, C = \frac{596}{585}$$

and

$$\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x = \left(\frac{4}{13}x^2 + \frac{32}{117}x + \frac{596}{585}\right)\sqrt[4]{(x-1)^5} + const$$

CONCLUSIONS

All integrals considered in the paper can also be evaluated by means of the "substitution" method. But using substitution to solve Example 3 by hand would become a tedious work, especially in the case of a higher order polynomial, so the method introduced in this paper can facilitate the process of integration of the herein given families of functions. There are other types of functions whose antiderivatives can be found without the "necessity" of integration. We will explore such functions in the upcoming paper. The use of the presented method is left to the reader in every particular case.

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