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Original Paper

The sum of the telescoping series formed by reciprocals of the cubic polynomials with three different negative integer roots

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ABSTRACT

This paper deals with the sum of a special telescoping series and is a free follow-up to author's preceding paper. The terms of this series are reciprocals of the cubic polynomial with three different negative integer roots. The main result of the paper is to derive a formula for the sum of this series. This formula uses the limit of the sequence of the partial sums and is expressed by harmonic numbers. After that the main result is verified by some examples using the basic programming language of the computer algebra system Maple 19.

KEYWORDS: sum of the series, sequence of partial sums, telescoping series, harmonic number, computer algebra system Maple

JEL CLASSIFICATION: I30

INTRODUCTION

Let us recall some basic terms. The series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

converges to a limit s if and only if the sequence of partial sums

$$s_n = a_1 + a_2 + \cdots + a_n$$

converges to s, i.e.

$$\lim_{n\to\infty} s_n = s.$$

We say that the series has a *sum* s and write

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$$\sum_{k=1}^{\infty} a_k = s.$$

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. The nth *harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

$H(0)$ being defined as 0. Basic and as well interesting information about harmonic numbers can be found in [1], [2]. For $n = 1,2, \dots ,10$ we get the following table:

$\boldsymbol{n}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{H(n)}$	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$

Table 1: First ten harmonic numbers [source: own modelling in Maple 19]

The *telescoping series* is any series where nearly every term cancels with a preceding or following term, so its partial sums eventually only have a fixed number of terms after cancellation. Interesting facts about telescoping series can be found in [3]. For example, the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)(k+4)}$$

has the general kth term, after partial fraction decomposition, in a form

$$a_k = \frac{1}{(k+1)(k+2)(k+4)} = \frac{1}{6}\left(\frac{2}{k+1} - \frac{3}{k+2} + \frac{1}{k+4}\right).$$

After that we arrange the terms of the nth partial sum $s_n = a_1 + a_2 + \cdots + a_n$ in a form where can be seen what is cancelling. Then we find the limit $\lim_{n\to\infty} s_n$ of the sequence of the partial sums s_n in order to find the sum s of the infinite telescoping series. In our case we get

$$s_n = \frac{1}{6}\left[\left(\frac{2}{2} - \frac{3}{3} + \frac{1}{5}\right) + \left(\frac{2}{3} - \frac{3}{4} + \frac{1}{6}\right) + \left(\frac{2}{4} - \frac{3}{5} + \frac{1}{7}\right) + \left(\frac{2}{5} - \frac{3}{6} + \frac{1}{8}\right) + \left(\frac{2}{6} - \frac{3}{7} + \frac{1}{9}\right) + \cdots\right.$$
$$\cdots + \left(\frac{2}{n-3} - \frac{3}{n-2} + \frac{1}{n}\right) + \left(\frac{2}{n-2} - \frac{3}{n-1} + \frac{1}{n+1}\right) + \left(\frac{2}{n-1} - \frac{3}{n} + \frac{1}{n+2}\right) +$$
$$\left. + \left(\frac{2}{n} - \frac{3}{n+1} + \frac{1}{n+3}\right) + \left(\frac{2}{n+1} - \frac{3}{n+2} + \frac{1}{n+4}\right)\right] =$$
$$= \frac{1}{6}\left(\frac{2}{2} - \frac{1}{3} - \frac{1}{4} - \frac{2}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}\right).$$

So we have

$$s = \lim_{n\to\infty} \frac{1}{6}\left(1 - \frac{1}{3} - \frac{1}{4} - \frac{2}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}\right) = \frac{1}{6}\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{5}{72} = 0.069\overline{4}.$$

PARTICULAR EXAMPLE

Let us consider a particular example – to determine the sum of the telescoping series formed by reciprocals of the cubic polynomial with three different specific negative integer roots.

Example 1 Using the nth partial sum calculate the sum $s(-2,-6,-9)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+6)(k+9)} \tag{1}$$

representing the telescoping series formed by reciprocals of the cubic polynomials with negative roots $k=-2$, $k=-6$, and $k=-9$.

By means of the CAS Maple 19 we get that the partial fraction decomposition of the kth term

$$a_k = \frac{1}{(k+2)(k+6)(k+9)}$$

has the form

$$a_k = \frac{1}{28(k+2)} - \frac{1}{12(k+6)} + \frac{1}{21(k+9)} = \frac{1}{84}\left(\frac{3}{k+2} - \frac{7}{k+6} + \frac{4}{k+9}\right).$$

The expression in parentheses we now express as the reciprocal of the natural numbers:

$$a_k = \frac{1}{84}\left(\frac{7-4}{k+2} - \frac{7}{k+6} + \frac{4}{k+9}\right) = \frac{7}{84}\left(\frac{1}{k+2} - \frac{1}{k+6}\right) - \frac{4}{84}\left(\frac{1}{k+2} - \frac{1}{k+9}\right),$$

i.e.

$$a_k = \frac{1}{12}\left(\frac{1}{k+2} - \frac{1}{k+6}\right) - \frac{1}{21}\left(\frac{1}{k+2} - \frac{1}{k+9}\right).$$

The nth partial sum $s_n(-2,-6,-9)$ of the infinite series (1) is

$$\begin{aligned} s_n(-2,-6,-9) &= \frac{1}{12}\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+6}\right) - \frac{1}{21}\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+9}\right) \\ &= \frac{1}{12} s_n(-2,-6) - \frac{1}{21} s_n(-2,-9)\,. \end{aligned}$$

Because

$$\begin{aligned} s_n(-2,-6) = {} & \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{n+1} + \frac{1}{n+2} - \\ & -\left(\frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \frac{1}{n+5} + \frac{1}{n+6}\right) = \\ & = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{n+3} - \frac{1}{n+4} - \frac{1}{n+5} - \frac{1}{n+6} \end{aligned}$$

and

$$s_n(-2,-9) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{n+1} + \frac{1}{n+2} -$$

$$-\left(\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+8}+\frac{1}{n+9}\right)=$$

$$=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}-\frac{1}{n+6}-\frac{1}{n+7}-\frac{1}{n+8}+\frac{1}{n+9},$$

we get

$$s_n(-2,-6,-9)=\frac{1}{12}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}-\frac{1}{n+6}\right)-$$

$$-\frac{1}{21}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{n+3}-\frac{1}{n+4}-\cdots-\frac{1}{n+8}-\frac{1}{n+9}\right).$$

Since for arbitrary integer a it holds

$$\lim_{n\to\infty}\frac{1}{n+a}=0\,,$$

we have

$$s(-2,-6,-9)=\lim_{n\to\infty}s_n(-2,-6,-9)\,,$$

so

$$s(-2,-6,-9)=\frac{1}{12}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)-\frac{1}{21}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}\right)=$$

$$=\frac{1}{12}[H(6)-H(2)]-\frac{1}{21}[H(9)-H(2)]=\frac{1}{12}\left(\frac{49}{20}-\frac{3}{2}\right)-\frac{1}{21}\left(\frac{7129}{2520}-\frac{3}{2}\right)=$$

$$=\frac{1}{12}\cdot\frac{19}{20}-\frac{1}{21}\cdot\frac{3349}{2520}=\frac{19}{240}-\frac{3349}{52920}=\frac{1681}{105840}\doteq 0.015882\,.$$

THE SUM OF THE TELESCOPING SERIES FORMED BY RECIPROCALS OF THE CUBIC POLYNOMIAL WITH THREE DIFFERENT NEGATIVE INTEGER ROOTS

This paper is a free follow-up to author's paper [4] dealing with the sum of the telescoping series formed by reciprocals of the quadratic polynomials with two different positive integer roots. Before we derive the main result of this paper, we present two lemmas.

Lemma 1 Let $A<B<C$ are positive natural numbers. Then a fraction

$$\frac{1}{(k+A)(k+B)(k+C)}$$

can be rewritten in the form

$$\frac{1}{(C-B)(C-A)(B-A)}\left(\frac{C-B}{k+A}-\frac{C-A}{k+B}+\frac{B-A}{k+C}\right). \tag{2}$$

Since $C-B=(C-A)-(B-A)$, we can (2) write in the form of the difference

$$\frac{1}{(C-B)(B-A)}\left(\frac{1}{k+A}-\frac{1}{k+B}\right)-\frac{1}{(C-B)(C-A)}\left(\frac{1}{k+A}-\frac{1}{k+C}\right). \tag{3}$$

Proof. The proof can be done in Maple 19 using the **simplify** command applied to expressions (2) and (3).

Lemma 2 Let $A < B$ are positive natural numbers. Then it holds

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+A}-\frac{1}{k+B}\right)=H(B)-H(A)\,, \tag{4}$$

where $H(n)$ is the nth harmonic number.

Proof. The sum $s(A,B)$ of the infinite series in (4) is the limit

$$\lim_{n\to\infty} s_n(A,B)$$

of the sequence of the partial sums

$$s_n(A,B)=\sum_{k=1}^{n}\left(\frac{1}{k+A}-\frac{1}{k+B}\right),$$

where

$$s_n(A,B)=\frac{1}{1+A}+\frac{1}{2+A}+\cdots+\frac{1}{B}+\frac{1}{1+B}+\frac{1}{2+B}+\cdots+\frac{1}{n+A-1}+\frac{1}{n+A}-$$
$$-\left(\frac{1}{1+B}+\frac{1}{2+B}+\cdots+\frac{1}{n+A-1}+\frac{1}{n+A}+\frac{1}{n+A+1}+\cdots+\frac{1}{n-1+B}+\frac{1}{n+B}\right)=$$
$$=\frac{1}{1+A}+\frac{1}{2+A}+\cdots+\frac{1}{B}-\frac{1}{n+A+1}-\cdots-\frac{1}{n-1+B}-\frac{1}{n+B}.$$

Hence we have

$$s(A,B)=\lim_{n\to\infty}\left(\frac{1}{1+A}+\frac{1}{2+A}+\cdots+\frac{1}{B}-\frac{1}{n+A+1}-\cdots-\frac{1}{n-1+B}-\frac{1}{n+B}\right)=$$
$$=\frac{1}{1+A}+\frac{1}{2+A}+\cdots+\frac{1}{B}=H(B)-H(A).$$

Now, let us consider the series formed by reciprocals of the cubic polynomial with three different negative integer roots $a > b > c$, i.e. the series

$$\sum_{k=1}^{\infty}\frac{1}{(k-a)(k-b)(k-c)} \tag{5}$$

and let us determine its sum $s(a,b,c)$.

Clearly, for arbitrary negative integers $a > b > c$ are $A=-a,\ B=-b,\ C=-c$ positive integers, $A<B<C$, so for $k=1,2,\ldots$ the expressions $k-a=k+A,\ k-b=k+B,\ k-c=k+C$ are positive integers, and we get the series

$$\sum_{k=1}^{\infty}\frac{1}{(k+A)(k+B)(k+C)} \tag{6}$$

and we determine its sum $s(A,B,C)$. We express the kth term a_k of this series as the sum of three partial fractions

$$a_k=\frac{1}{(k+A)(k+B)(k+C)}=\frac{P}{k+A}+\frac{Q}{k+B}+\frac{R}{k+C}\,.$$

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According to Lemma 1 we can write

$$a_k = \frac{1}{(C-B)(B-A)}\left(\frac{1}{k+A} - \frac{1}{k+B}\right) - \frac{1}{(C-B)(C-A)}\left(\frac{1}{k+A} - \frac{1}{k+C}\right),$$

so the nth partial sum is

$$s_n(A,B,C) = \frac{1}{(C-B)(B-A)}\sum_{k=1}^{n}\left(\frac{1}{k+A} - \frac{1}{k+B}\right) -$$
$$-\frac{1}{(C-B)(C-A)}\sum_{k=1}^{n}\left(\frac{1}{k+A} - \frac{1}{k+C}\right).$$

According to Lemma 2 the first partial sum equals $H(B) - H(A)$ and the second one equals $H(C) - H(A)$, so the sum $s(A,B,C)$ is

$$s(A,B,C) = \frac{H(B)-H(A)}{(C-B)(B-A)} - \frac{H(C)-H(A)}{(C-B)(C-A)}.$$

We have derived the following statement:

Theorem 1 The series

$$\sum_{k=1}^{\infty}\frac{1}{(k-a)(k-b)(k-c)},$$

where $a > b > c$ are negative integers, has the sum

$$s(a,b,c) = \frac{H(-b)-H(-a)}{(b-c)(a-b)} - \frac{H(-c)-H(-a)}{(b-c)(a-c)}, \tag{7}$$

where $H(n)$ is the nth harmonic number.

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(a,b,c)$ for $a = -1,-2,-3,-4$, for $b = a-1, a-2, \dots, a-5$ and for $c = b-1, b-2, \dots, b-5$. We use on the one hand an approximative direct evaluation of the sum

$$s(a,b,c,t) = \sum_{k=1}^{t}\frac{1}{(k-a)(k-b)(k-c)},$$

where $t = 10^7$, using the basic programming language of the computer algebra system Maple 19, and on the other hand the formula (7) for evaluation the sum $s(a,b,c)$. We compare 60 pairs of these ways obtained sums $s(a,b,c,t)$ and $s(a,b,c)$ to verify the formula (7). We use the following simple procedure `tsabcneg` and the following double repetition statement:

```
> tsabcneg:=proc(a,b,c,t)
      local A,B,C,k,sabc,sabct;
      A:=-a; B:=-b; C:=-c; sabct:=0;
      sabc:=(harmonic(B)-harmonic(A))/((B-C)*(A-B))
          -(harmonic(C)-harmonic(A))/(B-C)*(A-C));
      print("s(",a,b,c")=",evalf[12](sabc));
```

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```
      for k from 1 to t do
          sabct:=sabct+1/((k+A)*(k+B)*(k+C));
      end do;
      print("s(",a,b,c,t")=",evalf[12](sabct));
      print("diff=",evalf[12](abs(sabct-sabc)));
  end proc:
> for i from -1 by -1 to -4 do
      for j from i-1 by -1 to i-5 do
          for k from j-1 by -1 to j-5 do
              tsabcneg(i,j,k,10000000);
          end do;
      end do;
  end do;
```

The approximative values of the sums $s(a, b, c)$ rounded to 10 decimals obtained by this procedure are written into the table 1:

Table 1: Some approximate values of the sums $s(a, b, c)$ obtained by means

$\boldsymbol{a} = -\mathbf{1}$	$c = -3$	$c = -4$	$c = -5$	$c = -6$	$c = -7$
$b = -2$	0.0833333333	0.0694444444	0.0597222222	0.0525000000	00469047619.
$b = -3$	×	0.0555555556	0.0479166667	0.0422222222	0.0377976190
$b = -4$	×	×	0.0402777778	0.0355555556	0.0318783069
$b = -5$	×	×	×	0.0308333333	0.0276785714
$b = -6$	×	×	×	×	0.0245238095
$\boldsymbol{a} = -\mathbf{2}$	$c = -4$	$c = -5$	$c = -6$	$c = -7$	$c = -8$
$b = -3$	0.0416666667	0.0361111111	0.0319444444	0.0286904762	0.0260714286
$b = -4$	×	0.0305555556	0.0270833333	0.0243650794	0.0221726190
$b = -5$	×	×	0.0236111111	0.0212698413	0.0193783069
$b = -6$	×	×	×	0.0189285714	0.0172619048
$b = -7$	×	×	×	×	0.0155952381
$\boldsymbol{a} = -\mathbf{3}$	$c = -5$	$c = -6$	$c = -7$	$c = -8$	$c = -9$
$b = -4$	0.0250000000	0.0222222222	0.0200396825	0.0182738095	0.0168121693
$b = -5$	×	0.0194444444	0.0175595238	0.0160317460	0.0147652116
$b = -6$	×	×	0.0156746032	0.0143253968	0.0132054674
$b = -7$	×	×	×	0.0129761905	0.0119708995
$b = -8$	×	×	×	×	0.0109656085
$\boldsymbol{a} = -\mathbf{4}$	$c = -6$	$c = -7$	$c = -8$	$c = -9$	$c = -10$
$b = -5$	0.0166666667	0.0150793651	0.0137896825	0.0127182540	0.0118121693
$b = -6$	×	0.0134920635	0.0123511905	0.0114021164	0.0105985450
$b = -7$	×	×	0.0112103175	0.0103571429	0.0096340388
$b = -8$	×	×	×	0.0095039683	0.0088458995
$b = -9$	×	×	×	×	0.0081878307

Source: own modelling in Maple 19

Computation of 60 pairs of the sums $s(a, b, c)$ and $s(a, b, c, 10^7)$ took about 25 minutes. The absolute errors, i.e. the differences $|s(a, b, c) - s(a, b, c, 10^7)|$, are all only about $5 \cdot 10^{-15}$.

CONCLUSIONS

We dealt with the sum of the telescoping series formed by reciprocals of the cubic polynomials with three different negative integer roots $a > b > c$, i.e. with the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)(k-c)} .$$

We derived that the sum $s(a,b,c)$ of this series is given by the formula

$$s(a,b,c) = \frac{H(-b) - H(-a)}{(b-c)(a-b)} - \frac{H(-c) - H(-a)}{(b-c)(a-c)},$$

where $H(n)$ is the nth harmonic number.

We verified this result by computing 60 sums using the computer algebra system Maple 19. This series so belong to special types of infinite series, such as geometric series, which sums are given analytically by means of a simple formula.

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