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Original Paper

The universal solution of equations of balance of the transversely isotropic plate with initial stresses with slippery strength of the flat borders

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ABSTRACT

The topic of this paper is concentrated on the problem of mechanics of a deformed solid. In the first part we solved equilibrium equations of a transversal isotropic plate with initial stresses under mixed conditions on planar faces where we applied the method of decomposition of the sought functions into Fourier series by Legendre polynomials. Normal displacement and tangent voltage were assumed to be zero. In the second part we proposed a method of representing the general analytical solution of the obtained equilibrium equations.

KEY WORDS: transversely isotropic plate, initial stresses, equations of balance, universal solution

JEL CLASSIFICATION: C02, C30

INTRODUCTION

Initial stresses are widely used in solving problems of a formed solid [2, 3]. In [4, 5], a method for constructing equations of anisotropic shells and plates with initial (residual) stresses is outlined. It is based on the method of decomposition of sought functions into Fourier series by Legendre polynomials of thickness coordinate [8]. With respect to the coefficients of expansions, a system of differential equations and corresponding boundary conditions were obtained as a function of two independent variables. On this basis, in [6] a solution to the problem of the stress state of a transversal-isotropic plate with initial stresses weakened by a circular cylindrical cavity was found.

MATERIAL AND METHODS

The cavity surface and flat faces are free of external forces, and at infinity the plate is subject to constant tensile and shear forces. In this work, by the method of decomposition of the sought functions into Fourier series by Legendre polynomials, we derive the equation of the

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elastic equilibrium of a transversal-isotropic plate with initial stresses at the sliding finishing of plane faces (with zero of normal displacement and tangential stresses). The method of representing the general analytical solution of the obtained system of differential equations is presented.

RESULTS AND DISCUSSION

1 Equilibrium equations

Assume that the plate is related to the Cartesian coordinate system x_i $(i = 1,2,3)$, and it is located on the median plane S coinciding with the isotropy plane, and $x_3 \in [-h, h]$. The frontal boundary planes $x_3 = \pm h$ are slidably fixed, i.e.

$$u_3(x_1, x_2, \pm h) = 0, \ \sigma_{3\alpha}(x_1, x_2, \pm h) = 0 \ (\alpha = 1,2) \ , \tag{1.1}$$

and the boundary conditions on the cylindrical surface $\partial S \times [-h, h]$ are arbitrary.

For the problem, we use the method of decomposition of the components of the vector of displacements $u_j(x_1, x_2, x_3)$ into the Fourier series by Legendre polynomials $P_k(\zeta)$ of the thickness coordinate. Consider, given the boundary conditions (1.1), the components of the displacement vector in the form

$$u_\alpha = \sum_{k=0}^{N} u_\alpha^{(k)}(x) P_k(\varsigma), \ u_3 = \sum_{k=0}^{N} u_3^{(k)}(x)[P_k(\varsigma) - P_{k+2}(\varsigma)] \tag{1.2}$$

And we present the components of the stress tensor as follows

$$\sigma_{ij} = \sum_{k=0}^{N} \sigma_{ij}^{(k)}(x) P_k(\zeta), \tag{1.3}$$

where $x = (x_1, x_2) \in S, \ \varsigma = h^{-1}x_3 \in [-1,1], \ u_j^{(k)}(x), \ \sigma_{ij}^{(k)}(x)$ - coefficients of expansions, called moments (the moment number corresponds to the order of the Legendre polynomial), $N-$ is a natural number which we shall consider even $N = 2n$ $(n = 1,2,...\langle\infty)$. With respect to the coefficients of expansions $\sigma_{ij}^{(k)}, \ u_j^{(k)}$, a system of differential equations and corresponding boundary conditions is composed as a function of two independent variables. For a transversal isotropic plate, it splits into two independent groups of equations describing, respectively, symmetric and obliquely symmetric (relative to the median plane S) deformations of the plate. In symmetric deformation, taking into account boundary conditions (1.1), it has the form [6]

$$\begin{gathered} \partial_\alpha \sigma_{\alpha\beta}^{(2k)} - (4k+1)h^{-1} \sum_{s=1}^{k} \sigma_{3\beta}^{(2s-1)} = 0 \quad (\beta = 1,2; \ k = 0,1,...,n), \\ \partial_\alpha \sigma_{\alpha 3}^{(2k-1)} - (4k-1)h^{-1} \sum_{s=0}^{k-1} \sigma_{33}^{(2s)} + \frac{4k-1}{2h}(\sigma_{33}^{+} + \sigma_{33}^{-}) = 0 \quad (k = 1,...,n), \end{gathered} \tag{1.4}$$

where $\partial_\alpha = \partial / \partial x_\alpha, \ \sigma_{33}^{+}, \ \sigma_{33}^{-}-$ is the normal stress on the planes $x_3 = h, \ x_3 = -h$.

Consider a plate with a homogeneous field of initial stresses $P_{ij}^{(0)}$, and assume that $P_{ij}^{(0)} = const$ for $i = j$ and $P_{ij}^{(0)} = 0$, if $i \neq j$.

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Based on the equations [1]

$$\sigma_{ij} = c_{ijlm}\partial_\ell u_m + p_{il}\partial_\ell u_j \tag{1.5}$$

where c_{ijlm} – the elastic modulus tensor, we obtain, taking into account the expansions (1.2), the relations for a transversal-isotropic plate with a homogeneous field of initial stresses, hence

$$\begin{aligned}
&\sigma_{11} = \sum_{l=0}^{n}\left[\left(c_{11} + p_{11}^{(0)}\right)\varepsilon_{11}^{(2l)} + c_{12}\varepsilon_{22}^{(2l)} - (4l+1)c_{13}h^{-1}u_3^{(2l-1)}\right]P_{2l}(\zeta),\\
&\sigma_{22} = \sum_{l=0}^{n}\left[c_{12}\varepsilon_{11}^{(2l)} + \left(c_{11}\varepsilon_{11}^{(2l)} + p_{22}^{(0)}\right)\varepsilon_{22}^{(2l)} - (4l+1)c_{13}h^{-1}u_3^{(2l-1)}\right]P_{2l}(\zeta),\\
&\sigma_{33} = \sum_{l=0}^{n}\left[c_{13}\left(\varepsilon_{11}^{(2l)} + \varepsilon_{22}^{(2l)}\right) - (4l+1)(c_{33} + p_{33}^{(0)})h^{-1}u_3^{(2l-1)}\right]P_{2l}(\zeta),\\
&\sigma_{12} = \sum_{l=0}^{n}\left[\left(c_{66} + p_{11}^{(0)}\right)\varepsilon_{12}^{(2l)} + c_{66}\varepsilon_{21}^{(2l)}\right]P_{2l}(\zeta), \qquad \sigma_{21} = \sum_{l=0}^{n}\left[c_{66}\varepsilon_{12}^{(2l)} + \left(c_{66} + p_{22}^{(0)}\right)\varepsilon_{21}^{(2l)} + c_{12}\varepsilon_{22}^{(2l)}\right]P_{2l}(\zeta),\\
&\sigma_{13} = \sum_{l=1}^{n}\left[\left(c_{44} + p_{11}^{(0)}\right)\varepsilon_{13}^{(2l-1)} + c_{44}h^{-1}\underline{u}_1^{(2l)}\right]P_{2l-1}(\zeta), \ \ \sigma_{31} = \sum_{l=1}^{n}\left[c_{44}\varepsilon_{13}^{(2l-1)} + \left(c_{44} + p_{33}^{(0)}\right)h^{-1}\underline{u}_1^{(2l)}\right]P_{2l-1}(\zeta),\\
&\sigma_{23} = \sum_{l=1}^{n}\left[\left(c_{44} + p_{22}^{(0)}\right)\varepsilon_{23}^{(2l-1)} + c_{44}h^{-1}\underline{u}_2^{(2l)}\right]P_{2l-1}(\zeta), \ \ \sigma_{32} = \sum_{l=1}^{n}\left[c_{44}\varepsilon_{23}^{(2l-1)} + \left(c_{44} + p_{33}^{(0)}\right)h^{-1}\underline{u}_2^{(2l)}\right]P_{2l-1}(\zeta).
\end{aligned} \tag{1.6}$$

Here

$$\underline{u}_2^{(2l)} = (4l-1)\sum_{s=l}^{n}u_\alpha^{(2s)}, \quad \varepsilon_{\alpha\beta}^{(2l)} = \partial_\alpha u_\beta^{(2l)}, \ \varepsilon_{\alpha 3}^{(2l-1)} = \partial_\alpha (u_3^{(2l-1)} - u_3^{(2l-3)}),$$

$c_{11}, c_{12}, ..., c_{66}$ – two-index designations of elastic constants, i.e.

$c_{11} = c_{1111}$, $c_{12} = c_{1122}, ..., c_{66} = c_{1212}$. From relations (1.6) it follows that

$$\sigma_{33}^{+} + \sigma_{33}^{-} = 2\sum_{k=0}^{n}\left[c_{13}\left(\varepsilon_{11}^{(2k)} + \varepsilon_{22}^{(2k)}\right) - (4k+1)(c_{33} + p_{33}^{(0)})h^{-1}u_3^{(2k-1)}\right]. \tag{1.7}$$

Multiplying (1.6) by Legendre polynomials and integrating over the plate thickness, we obtain a relation connecting the moments of the stress components and the displacement vector, i.e.

$$\begin{aligned}
&\sigma_{11}^{(2k)} = \left(c_{11} + p_{11}^{(0)}\right)\varepsilon_{11}^{(2k)} + c_{12}\varepsilon_{22}^{(2k)} - (4k+1)c_{13}h^{-1}u_3^{(2k-1)},\\
&\sigma_{22}^{(2k)} = c_{12}\varepsilon_{11}^{(2k)} + \left(c_{11} + p_{22}^{(0)}\right)\varepsilon_{22}^{(2k)} - (4k+1)c_{13}h^{-1}u_3^{(2k-1)},\\
&\sigma_{33}^{(2k)} = c_{13}(\varepsilon_{11}^{(2k)} + \varepsilon_{22}^{(2k)}) - (4k+1)(c_{33} + p_{33}^{(0)})h^{-1}u_3^{(2k-1)},\\
&\sigma_{12}^{(2k)} = \left(c_{66} + p_{11}^{(0)}\right)\varepsilon_{12}^{(2k)} + c_{66}\varepsilon_{21}^{(2k)}; \ \sigma_{21}^{(2k)} = c_{66}\varepsilon_{12}^{(2k)} + \left(c_{66} + p_{22}^{(0)}\right)\varepsilon_{21}^{(2k)},\\
&\sigma_{13}^{(2k-1)} = \left(c_{44} + p_{11}^{(0)}\right)\varepsilon_{13}^{(2k-1)} + c_{44}h^{-1}\underline{u}_1^{(2k)}; \ \sigma_{31}^{(2k-1)} = c_{44}\varepsilon_{13}^{(2k-1)} + \left(c_{44} + p_{33}^{(0)}\right)h^{-1}\underline{u}_1^{(2k)},\\
&\sigma_{23}^{(2k-1)} = \left(c_{44} + p_{22}^{(0)}\right)\varepsilon_{23}^{(2k-1)} + c_{44}h^{-1}\underline{u}_2^{(2k)}; \ \sigma_{32}^{(2k-1)} = c_{44}\varepsilon_{23}^{(2k-1)} + \left(c_{44} + p_{33}^{(0)}\right)h^{-1}\underline{u}_2^{(2k)}.
\end{aligned} \tag{1.8}$$

Substituting (1.7), (1.8) into equations (1.4), we obtain such a system of equations

$$\left(c_{66}+p_{11}^{(0)}\right)\frac{\partial^2 u_\alpha^{(2k)}}{\partial x_1^2}+\left(c_{66}+p_{22}^{(0)}\right)\frac{\partial^2 u_\alpha^{(2k)}}{\partial x_2^2}+\left(c_{12}+c_{66}\right)\frac{\partial \ell^{(2k)}}{\partial x_\alpha}-$$
$$-\frac{4k+1}{n}\left[\left(c_{13}+c_{44}\right)\frac{\partial u_3^{(2k-1)}}{\partial x_\alpha}+\frac{c_{44}+p_{33}^{(0)}}{h}\sum_{s=1}^{n}\beta_{2s}^{(k)}u_\alpha^{(2s)}\right]=0\ \left(\alpha=1,2;\ k=\overline{0,n}\right); \quad (1.9)$$

$$\left(c_{44}+p_{11}^{(0)}\right)\frac{\partial^2\left(u_3^{(2k-1)}-u_3^{(2k-3)}\right)}{\partial x_1^2}+\left(c_{44}+p_{22}^{(0)}\right)\frac{\partial^2\left(u_3^{(2k-1)}-u_3^{(2k-3)}\right)}{\partial x_2^2}+$$
$$+\frac{4k-1}{n}\left[\left(c_{13}+c_{44}\right)\sum_{s=k}^{n}\ell^{(2s)}-\frac{c_{33}+p_{33}^{(0)}}{h}\sum_{s=k}^{n}(4s+1)u_3^{(2s-1)}\right]=0\ \left(k=\overline{1,n}\right), \quad (1.10)$$

where $\ell^{(2k)}=\partial u_1^{(2k)}/\partial x_1+\partial u_2^{(2k)}/\partial x_2$, $\beta_{2s}^{(k)}-$ absolute constant.

$$\beta_{2s}^{(k)}=\begin{cases} s(2s-1)\ ,1\le s\le k;\\ k(2k-1)\ ,k\le s\le n.\end{cases} \quad (1.11)$$

$$\left(c_{44}+p_{11}^{(0)}\right)\frac{\partial^2 u_3^{(2k-1)}}{\partial x_1^2}+\left(c_{44}+p_{22}^{(0)}\right)\frac{\partial^2 u_3^{(2k-1)}}{\partial x_2^2}+\sum_{s=1}^{n}\beta_{2s}^{(k)}[\left(c_{13}+c_{44}\right)\ell^{(2s)}-$$
$$-\frac{(4s+1)(c_{33}+p_{33}^{(0)})}{h}u_3^{(2s-1)}]=0. \quad (1.12)$$

Assuming $p_{11}^{(0)}=p_{22}^{(0)}$ and introducing complex variables $z=x_1+ix_2$, $\overline{z}=x_1-ix_2$, we write equations (1.9) and (1.12) in this way

$$c_{66}^{\bullet}\Delta u_+^{(2k)}+2(c_{12}+c_{66})\partial_{\overline{z}}\ell^{2k}-(4k+1)h^{-1}[2c^{\bullet}c_{44}\partial_{\overline{z}}u_3^{(2k-1)}$$
$$+\widetilde{c}_{44}h^{-1}\sum_{s=1}^{n}\beta_{2s}^{(k)}u_+^{(2s)}]=0\quad \left(k=\overline{0,n}\right), \quad (1.13)$$

$$c_{44}^{\bullet}\Delta u_3^{(2k-1)}+h^{-1}\sum_{s=1}^{n}\beta_{2s}^{(k)}[c^{\bullet}c_{44}\ell^{(2s)}-(4s+1)\widetilde{c}_{33}u_3^{(2s-1)}]=0\quad \left(k=\overline{1,n}\right), \quad (1.14)$$

Here $\Delta=4\partial_z\partial_{\overline{z}}-$ is the Laplace operator, $2\partial_z=\partial/\partial x_1-i\partial/\partial x_2$, $2\partial_{\overline{z}}=\partial/\partial x_1+i\partial/\partial x_2$.

$$u_+^{(2k)}=u_1^{(2k)}+iu_2^{(2k)},\ \ell^{(2k)}=\partial_z u_+^{(2k)}+\partial_{\overline{z}}\overline{u}_+^{(2k)}, \quad (1.15)$$
$$c_{44}^{\bullet}=c_{44}+p_{11}^{(0)},\ c_{66}^{\bullet}=c_{66}+p_{11}^{(0)},\ \widetilde{c}_{44}=c_{44}+p_{33}^{(0)},\widetilde{c}_{33}=c_{33}+p_{33}^{(0)},c^{\bullet}=\left(c_{13}+c_{44}\right)/c_{44}.$$

2 General analytical solution

We present a method for representing a general analytical solution to the system of equations (1.13), (1.14). We write equalities (1.14) in the form

$$\sum_{s=1}^{n}\beta_{2s}^{(k)}\ell^{(2s)}=b_k\ \left(k=\overline{1,n}\right), \quad (2.1)$$

where

$$b_k = -\frac{c_{44}^{\bullet} h}{c^{\bullet} c_{44}} \Delta u_3^{(2k-1)} + \frac{\widetilde{c}_{33}}{c^{\bullet} c_{44} h} \sum_{s=1}^{n} (4s+1) \beta_{2s}^{(k)} u_3^{(2s-1)} \tag{2.2}$$

and define the values of the functions

$$\ell^{(2k)} = \frac{1}{4k-1}\left(b_k - b_{k-1}\right) - \frac{1}{4k+3}\left(b_{k+1} - b_k\right) \ \left(k = 1, n-1\right),$$
$$\ell^{(2n)} = \frac{1}{4n-1}\left(b_n - b_{n-1}\right). \tag{2.3}$$

From here, taking into account expression (2.2), we find

$$\ell^{(2k)} = \frac{c_{44}^{\bullet} h}{c^{\bullet} c_{44}} [\frac{1}{4k+3} \Delta(u_3^{(2k+1)} - u_3^{(2k-1)}) - \frac{1}{4k-1} \Delta(u_3^{(2k-1)} - u_3^{(2k-3)})] + \frac{(4k+1)\widetilde{c}_{33}}{c^{\bullet} c_{44} h} u_3^{(2k-1)};$$
$$\ell^{(2n)} = -\frac{c_{44}^{\bullet} h}{(4n-1) c^{\bullet} c_{44}} \Delta(u_3^{(2n-1)} - u_3^{(2n-3)}) + \frac{(4n+1)\widetilde{c}_{33}}{c^{\bullet} c_{44} h} u_3^{(2n-1)}. \tag{2.4}$$

We apply the operation ∂_z to equation (1.13) and in the resulting equality we consider the real part. Taking into account the formula (1.15), we obtain

$$c_{11}^{\bullet} \Delta \ell^{(0)} = 0 \quad (k = 0), \tag{2.5}$$

$$c_{11}^{\bullet} \Delta \ell^{(2k)} - (4k+1) h^{-1} [c^{\bullet} c_{44} \Delta u_3^{(2k-1)} + \widetilde{c}_{44} h^{-1} \sum_{s=1}^{n} \beta_{2s}^{(k)} \ell^{2s}] = 0 \quad (k = 1, n). \tag{2.6}$$

It follows from (2.5) that

$$\mathrm{c}_{66} \ell^{(0)} = æ_e \mathrm{u}, \tag{2.7}$$

where $u -$ arbitrary harmonic function, $æ_e^{\bullet} = 2\mathrm{c}_{66}^{\bullet} / (\mathrm{c}_{12} + \mathrm{c}_{66})$.

Equalities (2.6), taking into account the values of (2.4), are transformed to

$$\frac{1}{4k+3} \Delta\Delta u_3^{(2k+1)} - \frac{2(4k+1)}{(4k-1)(4k+3)} \Delta\Delta u_3^{(2k-1)} + \frac{1}{4k-1} \Delta\Delta u_3^{(2k-3)} + \frac{(4k+1) a \widetilde{c}_{33}}{c_{11}^{\bullet} h^2} \Delta u_3^{(2k-1)} -$$
$$- \frac{(4k+1)\widetilde{c}_{33}\widetilde{c}_{44}}{c_{11}^{\bullet} c_{44}^{\bullet} h^4} \sum_{s=1}^{n} (4s+1) \beta_{2s}^{(k)} u_3^{(2s-1)} = 0 \ \left(k = 1,\ n-1\right);$$
$$\frac{1}{4n-1} \Delta\Delta(u_3^{(2n-1)} - u_3^{(2n-3)}) - \frac{(4n+1)\widetilde{c}_{33}}{c_{11}^{\bullet} h^2} [a \Delta u_3^{(2n-1)} - \frac{\widetilde{c}_{44}}{c_{44}^{\bullet} h^2} \sum_{s=1}^{n} (4s+1) \beta_{2s}^{(n)} u_3^{(2s-1)}] = 0 \ \left(k = n\right). \tag{2.8}$$

Introducing the notation, $c_{66} u_3^{(2k-1)} = u_k \quad (k = 1,2,...,n)$, we represent the system (2.8) in standard form in this way

$$\sum_{k=1}^{n} L_{pk}(\Delta) u_k = 0 \ \left(p = 1,2,...,n\right), \tag{2.9}$$

where $L_{pk}(\Delta)$ – differential operators of the form

$$L_{pk}(\Delta) = \alpha_{pk} h^4 \Delta\Delta + \beta_{pk} h^2 \Delta + \gamma_{pk} \tag{2.10}$$

$\alpha_{pk}, \beta_{pk}, \gamma_{pk}$ – dimensionless constants whose explicit expressions are easy to write out.

To solve the system of equations (2.9), we use the operator method [7]. Consider the characteristic equation

$$\det\left\|\alpha_{pk}k^2 + \beta_{pk}k + \gamma_{pk}\right\| = 0 \tag{2.11}$$

And we will assume that it has simple, non-zero roots k_m $(m = 1,2,...,2n)$. Then, using the same method [6], we find

$$c_{66}u_3^{(2k-1)} = \sum_{m=1}^{2n} c_m^{(2k-1)} V_m, \tag{2.12}$$

where V_m – meta-harmonic functions satisfying the equalities

$$\Delta V_m - k_m h^{-2} V_m = 0, \tag{2.13}$$

and $c_m^{(2k-1)}$ – constants defined by algebraic complements of elements of some line of the determinant

$$\left|\alpha_{pk}k_m^2 + \beta_{pk}k_m + \lambda_{pk}\right|_{n\times n}.$$

According to (2.12), the moments of deformations (2.4) take the form

$$c_{66}he^{(2k)} = \sum_{m=1}^{2n} c_m^{(2k)} V_m, \tag{2.14}$$

where

$$c_m^{(2k)} = \frac{c_{44}^{\bullet}k_m}{c^{\bullet}c_{44}}\left[\frac{1}{4k+3}c_m^{(2k+1)} - \frac{2(4k+1)}{(4k-1)(4k+3)}c_m^{(2k-1)} + \frac{1}{4k-1}c_m^{(2k-3)}\right] + \frac{(4k+1)\tilde{c}_{33}}{c^{\bullet}c_{44}}c_m^{(2k-1)};$$
$$c_m^{(2n)} = -\frac{c_{44}^{\bullet}k_m}{(4n-1)c^{\bullet}c_{44}}\left(c_m^{(2n-1)} - c_m^{(2n-3)}\right) + \frac{(4n+1)\tilde{c}_{33}}{c^{\bullet}c_{44}}c_m^{(2n-1)}. \tag{2.15}$$

If we take a harmonic function u in the form of the real part of some harmonic function $\varphi'(z)$ (the prime denotes the derivative with respect to the variable), i.e.
$u = \varphi'(z) + \overline{\varphi'(z)}$ and take into account formula (1.15), then equalities (2.7) and (2.14) can be represented in this way

$$c_{66}\left(\partial_z u_+^{(0)} + \partial_{\bar{z}} \overline{u}_+^{(0)}\right) = æ_e\left[\varphi'(z) + \overline{\varphi'(z)}\right],$$
$$c_{66}\left(\partial_z u_+^{(2k)} + \partial_{\bar{z}} \overline{u}_+^{(2k)}\right) = \frac{h}{2}\sum_{m=1}^{2n} a_m^{(2k)} \Delta V_m \quad (k = 1, n). \tag{2.16}$$

From here we find the moments of the components of the displacement vector

$$c_{66}u_+^{(0)} = æ_{ee}^{\bullet}\varphi(z) + ih\partial_{\bar{z}} y_0,$$
$$c_{66}u_+^{(2k)} = h\sum_{m=1}^{2n} a_m^{(2k)} \partial_{\bar{z}} V_m + ih\partial_{\bar{z}} y_k \quad (k = 1, n), \tag{2.17}$$

where $a_m^{(2k)} = 2k_m^{-1} c_m^{(2k)}$, y_k – arbitrary, sufficiently smooth real functions. They must be chosen so that equations (1.13) are satisfied. Therefore, if we introduce in (1.13) the values of the moments (2.7), (2.12), (2.14) and (2.17), we obtain the equalities

$$\partial_{\bar{z}} \Delta y_0 = 4ih^{-1}\overline{\varphi''(z)}, \tag{2.18}$$

$$\partial_{\bar{z}} U_k = 0 \quad \left(k = \overline{1,n}\right), \tag{2.19}$$

where $U_k = X_k + iY_k$ – complex function, the real part of which is a linear combination of metagarmonic functions V_m, i.e.

$$X_k = c_0 \sum_{m=1}^{2n} O_m^{(2k)} V_m \quad \left(c_0 = c_{11}^{\bullet} / 2c_{66}h\right), \tag{2.20}$$

and the imaginary part is determined by the formula

$$Y_k = \Delta y_k - \frac{(4k+1)\widetilde{c}_{44}}{c_{66}h^2} \sum_{s=1}^{n} \beta_{2s}^{(2k)} y_s \ . \tag{2.21}$$

Here

$$O_m^{(2k)} = a_m^{(2k)} k_m - \frac{(4k+1)}{c_{11}^{\bullet}} \left[2c^{\bullet} c_{44} c_m^{(2k-1)} + \widetilde{c}_{44} \sum_{s=1}^{n} \beta_{2s}^{(k)} a_m^{(2s)} \right]. \tag{2.22}$$

It is easy to verify that the constants $O_m^{(2k)}$ are identically equal to zero for $\forall k \in [1, n]$. This follows from the fulfillment of equalities (2.6), taking into account the linear independence of meta-harmonic functions V_m. According to (2.18), we have

$$y_0 = ih^{-1}\left[z\overline{\varphi(z)} - \bar{z}\varphi(z) + \overline{\psi_0(z)} - \psi_0(z)\right], \tag{2.23}$$

where $\psi_0(z)$ – arbitrary holomorphic function; equation (2.19) after integration over the variable is reduced to the equality

$$U_k = f_k(z), \tag{2.24}$$

in which arbitrary analytic functions $f_k(z)$ are denoted. It follows from the last equality that the real part U_k should be a harmonic function, and since it is identically equal to zero, then $\mathrm{Re}[f_k(z)] = 0$, therefore, $f_k = ic_k$. Given that the functions y_k are determined up to constant terms, we can set the constants c_k to zero. Thus, from the equalities $\mathrm{Y_k} = 0$ we obtain the system of equations, which we write in the standard form this way

$$\sum_{l=1}^{n} \left(q_{kl} - \delta_{kl} h^2 \Delta\right) y_l = 0 \ \left(k = \overline{1,n}\right), \tag{2.25}$$

where δ_{kl} - Kronecker symbol, $q_{kl} = (4k+1)\beta_{2l}^{(k)} \widetilde{c}_{44} / c_{66}$.

Consider the characteristic equation

$$\det\left\| q_{kl} - \lambda \delta_{kl} \right\| = 0 \tag{2.26}$$

and assume that it has simple and non-zero roots λ_s. Then, by the above method, we find functions y_k, i.e.

$$y_k = \sum_{s=1}^{n} b_s^{(2k)} w_s, \tag{2.27}$$

where w_s – meta-harmonic functions satisfying the equalities

$$\Delta w_s - \lambda_s h^{-2} w_s = 0, \tag{2.28}$$

constants $b_s^{(2k)}$ are determined by algebraic complements of elements of some line of the determinant

$$\left| q_{kl} - \lambda_s \delta_{kl} \right|_{n\times n}.$$

According to formulas (2.23) and (2.27), the moments of the variable $u_+^{(2n)}$ take the form

$$\begin{aligned} c_{66}u_+^{(0)} &= æ^{\bullet}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}; \\ c_{66}u_+^{(2k)} &= h\sum_{m=1}^{2n} a_m^{(2k)} \partial_{\bar z} V_m + ih\sum_{s=1}^{n} b_s^{(2k)} \partial_{\bar z} w_s, \end{aligned} \tag{2.29}$$

where $\overline{\psi(z)} = \overline{\psi'_{\bullet}(z)}$, $æ^{\bullet} = 1 + æ_e^{\bullet}$.

Thus, the values of functions (2.12), (2.14) and (2.29), together with equalities (2.13) and (2.28), constitute the general solution of the system of equations (1.13), (1.14).

CONCLUSIONS

We applied method of decomposition of unknown functions into Fourier series, in the Legendre exposition of polynomials we took into account equations of equilibrium elasticity of the transversely isotropic plate with initial stresses at mixed conditions on the flat borders. We supposed the normal transference and touch stresses equal to zero. We proposed method of presentation of universal analytic solution of received equations. Found solution allows describing the stress state on the surface of a transversally isotropic plate.

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