{MERAA} Mathematics in Education, Research and Applications ISSN 2453-6881

Math Educ Res Appl, 2019(5), 2, 53-60

Received: 2019-09-30
Accepted: 2019-11-22
Online published: 2019-12-01
DOI: https://doi.org/10.15414/meraa.2019.05.02.53-60

Original Paper

Development and prospects of Stewart's theorem research

Olena Melnichenko[*], Uliana Revytska

Bila Tserkva National Agrarian University, Department of Mathematics and Physics, Ukraine

ABSTRACT

This paper is devoted to the study of Stewart's theorem, its consequences, development and prospects of research of the theorem under consideration. The paper focuses on the Diophantine equations and their relation to the Stewart's theorem. The problem of determination of integer solutions of the Diophantine equations was considered, and some modern researches of the Stewart's theorem are presented from the point of view of finding integer solutions of it, which are related to the first and second order Diophantine equations. The well-known integer solutions of the Stewart's theorem, and the definitions, which have been formulated in the form of a table, are presented in this paper. Some practical applications of the Stewart's theorem focused on computing the length of a segment, that connects the vertex of a triangle with its inner point, are relevant in the area of logistics, management and designing.

KEYWORDS: triangle geometry, Stewart's theorem, Diophantine equations, integer solutions

JEL CLASSIFICATION: C02, C30

INTRODUCTION

Stewart's theorem is one of the classical problems of triangle geometry and is partially represented in the elementary geometry educational literature [7, 11]. The most well-known problem-consequences of the Stewart's theorem are formulas for calculating the lengths of the medians and bisectors of a triangle on its sides [2]. The consequence of the Stewart's theorem is the Ptolemy theorem, and the Apollonian theorem that is the partial case of Ptolemy theorem [8]. We have systematized problems-consequences and proofs of the theorem and set task to move away from the classical method of the topic presentation and to characterize modern directions of researches of the theorem and to find its relations with other sections of mathematics. The question of finding an integer solution of the Stewart equality as a search for the solution of the Diophantine equation remains interesting.

[*] Corresponding author: Assoc. prof. Olena Melnichenko, PhD., Bila Tserkva National Agrarian University, Department of Mathematics and Physics, pl. 8/1 Soborna, Bila Tserkva, Kyivska oblast, 09117 Ukraine, E-mail: mela731@ukr.net

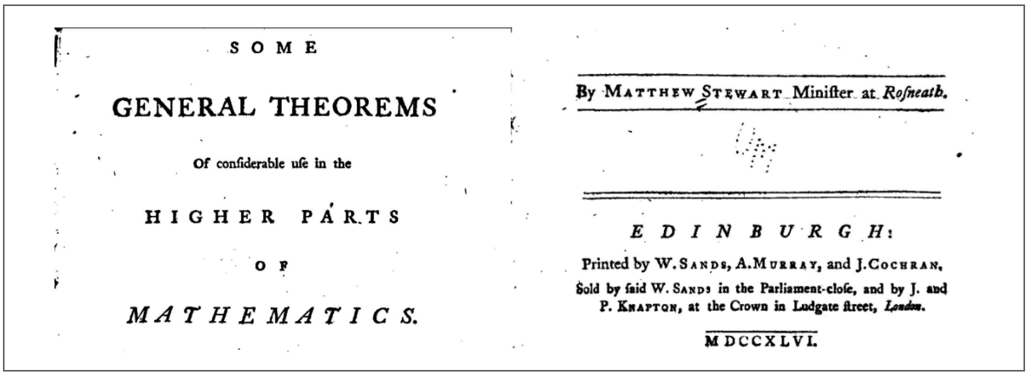
MATERIAL AND METHODS

The purpose of the paper is to formulate the Stewart's theorem and its development, to present modern studies of the theorem under consideration. We refer the integer solutions of the Stewart's theorem and show the relation to the Diophantine equations.

The research methods used in this paper are universally recognized methods of scientific knowledge [10]:
- Theoretical: study and analysis of relevant scientific literature, textbooks and materials of electronic publications;
- Inductive: collection, systematization and classification of existing proofs of the Stewart's theorem, its consequences and current studies of the theorem under consideration;
- Practical: application of theoretical knowledge and practical skills to create problems and their solutions.

RESULTS AND DISCUSSION

Scottish mathematician Stewart Matthew (1717 − 1785) published a scientific paper in 1746 (Fig. 1) in which he presented his theorem with substantiation [15]. This theorem is called Stewart's theorem, and is expressed by (1) and displayed in Fig. 2.


SOME
GENERAL THEOREMS
Of confiderable ufe in the
HIGHER PARTS
OF
MATHEMATICS.

By MATTHEW STEWART Minifter at Rofneath.

EDINBURGH:
Printed by W. SANDS, A. MURRAY, and J. COCHRAN,
Sold by faid W. SANDS in the Parliament-clofe, and by J. and P. KNAPTON, at the Crown in Ludgate ftreet, London.
MDCCXLVI.

Figure 1

Stewart's theorem [20]. Let ΔABC be an arbitrary triangle (Fig. 2). For any point *D* on the side of the BA the following formula holds:

$$CD^2 = CB^2 \cdot \frac{AD}{AB} + AC^2 \cdot \frac{BD}{AB} - BD \cdot AD \quad \text{or} \quad d^2 = a^2 \cdot \frac{n}{c} + b^2 \cdot \frac{m}{c} - n \cdot m \qquad (1)$$

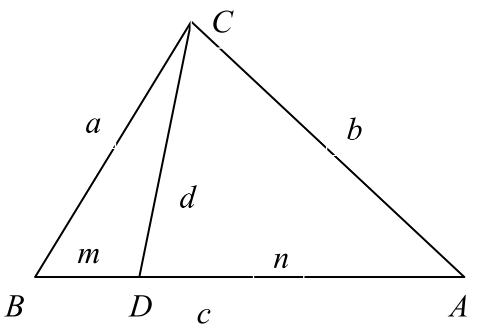


Figure 2

There are various proofs of Stewart's theorem: by Pythagorean Theorem, coordinate method, cosine theorem, vector method [1, 2, 8]. The work [2] presents the consequences of the Stewart's theorem: finding the length of a segment with ends on the sides of a triangle with possible boundary cases, and finding the distance from the vertex to the inner point of the triangle. As a development of the Stewart's theorem Willie Yong and Jim Bound in their paper Using Stewart's Theorem [20] presents the solution of the following problem:

Task 1. If in a triangle ΔABC side $CB > CA$ and segment CT belongs to the bisector of the angle $\angle SCA$, moreover $BS = TA$, points S and T belong to the BA side (Fig. 3). Then:

$$CS^2 - CT^2 = (CB - CA)^2 .$$

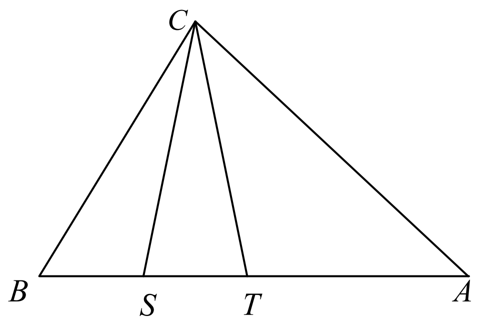


Figure 3

Substantiation. Known: $BT + TA = AB$.
By the property of bisector CT:
$\frac{BT}{TA} = \frac{CB}{CA}$, or $\frac{BT}{BA - BT} = \frac{CB}{CA}$.

Therefore, $BT = \frac{BA \cdot CB}{CA + CB}$, $TA = BA - BT = \frac{BA \cdot CB}{CA + CB}$. (2)

Consider that $BS = TA$, by the Stewart's theorem we have:

$$CT^2 = CB^2 \cdot \frac{AT}{AB} + AC^2 \cdot \frac{BT}{AB} - BT \cdot AT, \quad CS^2 = CB^2 \cdot \frac{BT}{AB} + AC^2 \cdot \frac{AT}{AB} - AT \cdot BT . \tag{3}$$

Substituting (3) into (2), we get: $CS^2 = \frac{CB^3 + AT}{AC + CB} - \frac{BA^2 \cdot CB \cdot AC}{(AC + CB)^2}$, i.e.,

$$CS^2 = CB^2 - CB \cdot CA + CA^2 - \frac{BA^2 \cdot CB \cdot AC}{(AC + CB)^2} . \tag{4}$$

In view of (3), (4) we subtract and obtain:

$$CS^2 - CT^2 = CB^2 - 2 \cdot CB \cdot CA + CA^2 = (CB - CA)^2 .$$

One of the problems that were formulated by Willie Yong and Jim Bound is presented below [16].

Problem 1. Prove that in the right triangle the sum of the squares of distances from the vertex of the right angle to the three points of hypotenuse is equal to $\frac{5}{9}$ of the length of the hypotenuse squared, i.e., $CS^2 + CT^2 = \frac{5}{9} \cdot AB$. The solution was presented in [17].

Solution. In view of the Stewart's theorem the following equalities hold (Fig. 4):

{MERAA} Mathematics in Education, Research and Applications ISSN 2453-6881
Math Educ Res Appl, 2019(5), 2, 53-60

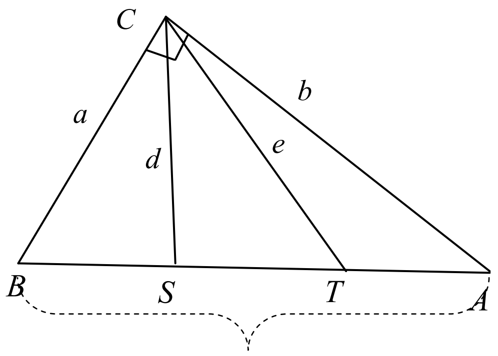


Figure 4

$$\begin{cases} c^2\cdot\left(\frac{2}{3}a\right)+b^2\cdot\left(\frac{1}{3}a\right)=a\cdot\left(d^2+\left(\frac{2}{3}a\right)\cdot\left(\frac{1}{3}a\right)\right), \\ c^2\cdot\left(\frac{2}{3}a\right)+b^2\cdot\left(\frac{2}{3}a\right)=a\cdot\left(e^2+\left(\frac{2}{3}a\right)\cdot\left(\frac{1}{3}a\right)\right). \end{cases}$$

We divide the right hand sides of the above equations with a and get

$$\begin{cases} \frac{2}{3}c^2+\frac{1}{3}b^2=d^2+\left(\frac{2}{3}a\right)\cdot\left(\frac{1}{3}a\right), \\ \frac{1}{3}c^2+\frac{2}{3}b^2=e^2+\left(\frac{2}{3}a\right)\cdot\left(\frac{1}{3}a\right). \end{cases}$$

Adding the last two equations results in: $c^2+b^2=d^2+e^2+\frac{4}{9}\cdot a$.

According to the Pythagorean theorem we have: $d^2+e^2=\frac{5}{9}\cdot a$. What was to be shown.

Bretschneider's formula describes the relation among the elements of the quadrilateral *ABCD* and is related to the Stewart’s theorem.
Let us denote (Fig. 5) $AB=a$, $DC=b$, $CD=c$, $AD=d$, $AC=e$, $BD=f$ [19].

Theorem (Bretschneider's formula). The following formula holds

$$(e\cdot f)^2=(a\cdot c)^2+(b\cdot d)^2-2\cdot a\cdot b\cdot c\cdot d\cdot\cos(\angle A+\angle C).$$

Proof. We construct two triangles *ABF* and *ADE*, which are similar to the triangles *CAD* and *CAB* respectively.
The similarty of the triangles yields to
$\frac{AF}{a}=\frac{c}{e}$, $\frac{BF}{a}=\frac{d}{c}$, $\frac{AE}{b}=\frac{d}{e}$, $\frac{DE}{b}=\frac{a}{e}$.
Thus, $AF=\frac{ac}{e}$, $BF=\frac{ad}{c}$, $AF=\frac{bd}{e}$, $DE=\frac{ad}{e}$.
The sum of vertex angles *D* and *E* of quadrilateral *BDEF* equals the sum of angles of *ΔABD*, i.e., is *180*°.
Thus, the lines *BF* and *DE* are parallel.

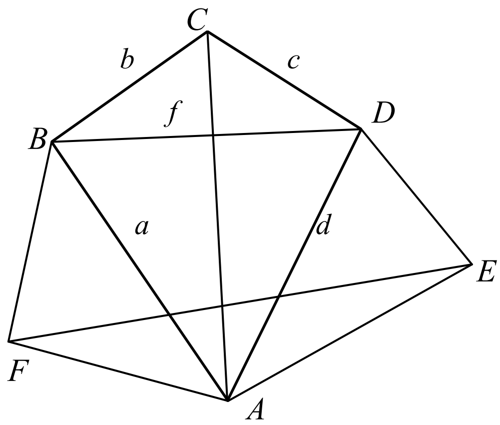


Figure 5

Since *BF* = *DE*, the quadrilateral *BDEF* is a parallelogram, *FE* = *BD* = *f*.
The angle *∠EAF* of triangle *ΔAEF* equals to the sum of *∠A* and *∠C* (from construction), in other words *∠EAF* = *∠A* +*∠C*. Law of cosines for *ΔAEF* implies that

$$(f)^2=\left(\frac{a\cdot c}{e}\right)^2+\left(\frac{b\cdot d}{e}\right)^2-\frac{2\cdot a\cdot b\cdot c\cdot d}{e}\cdot\cos(\angle A+\angle C).$$

Stewart’s theorem can be considered as the consequence of Bretschneider's formula, in case of degenerate quadrilateral *ABCD*, i.e., when the point *D* belongs to *AC*.

Let us consider the notion of Diophantine equations and its relation with the Stewart's theorem. A Diophantine equation is a polynomial equation, usually in two or more unknowns, such that only the integer solutions are sought or studied. It has to be mentioned that the main problems of study of Diophantine equations are existence of its solutions and also the existence of an algorithm for solving such equations [12, 13]. The problem of finding all integer solutions of Diophantine equations was quite popular among mathematicians. For example, the study of existence of integer solutions of Diophantine equations in the form $ax^2+bxy+cy^2=d$, where a, b, c, d are arbitrary integer numbers, were presented in [9, 18].

In [3] Keskin considered the Diophantine equations of the form $x^2-L_nxy+(-1)^n y^2=\pm 5^\gamma$, which under the assumptions $n > 0$ and $r > 1$ have the integer solutions that are Fibonacci numbers or Lucas numbers.

The authors Keskin and Yosma [6] considered the Diophantine equations of the following form: $x^2-5F_nxy+5\cdot(-1)^n y^2=\pm 5^\gamma$, with $n > 1$ and $r > 0$. It was also highlighted that for all $n\geq 2$ the Fibonacci-Lucas sequences are defined:

$$F_n=F_{n-1}+F_{n-2},\ F_0=0, F_1=1,\ L_n=L_{n-1}+L_{n-2},\ L_0=2, L=1.$$

Besides, the Fibonacci and Lucas numbers are defined for negative indices, for all $n\geq 1$, $F_{-n}=(-1)^{n+1}F_n$, $L_{-n}=(-1)^{n+1}L_n$ [4, 5, 14].

There is a relation between the Stewart's theorem and Diophantine equations, thus for finding integer solutions of Stewart's equation one has to apply the known algorithms for solving Diophantine equations. Consider the finding of integer solutions of the Stewart's equation of the form $\frac{b^2}{a}\cdot x+\frac{c^2}{a}\cdot y-xy=p^2$. To the right- and left-hand sides we add $-\frac{b^2}{a}\cdot\frac{c^2}{a}$.

We obtain: $\frac{b^2}{a}\cdot x+\frac{c^2}{a}\cdot y-xy-\frac{b^2}{a}\cdot\frac{c^2}{a}=p^2-\frac{b^2}{a}\cdot\frac{c^2}{a}$.

We factorize the equation: $\frac{b^2}{a}\cdot\left(x-\frac{c^2}{a}\right)-y\cdot\left(x-\frac{c^2}{a}\right)=p^2-\frac{b^2}{a}\cdot\frac{c^2}{a}$

and get: $\left(\frac{b^2}{a}-y\right)\cdot\left(x-\frac{c^2}{a}\right)=p^2-\frac{b^2}{a}\cdot\frac{c^2}{a}$.

In case $p^2-\frac{b^2}{a}\cdot\frac{c^2}{a}=0$, we have that one of the factors is zero.

In other words $\left(\frac{b^2}{a}-y\right)=0$ or $\left(x-\frac{c^2}{a}\right)=0$.

Let $\left(x-\frac{c^2}{a}\right)=0$, then we choose the value so that c^2 could be completely divided by a.

This implies $\left(x-\frac{c^2}{a}\right)=0$ and thus we compute $x=\frac{c^2}{a}$.

{MERAA} Mathematics in Education, Research and Applications ISSN 2453-6881
Math Educ Res Appl, 2019(5), 2, 53-60

We find b^2 such that $\frac{b^2}{a}\cdot\frac{c^2}{a}$ is a square of some integer and is equal to p^2, this could be rewritten as $p^2-\frac{b^2}{a}\cdot\frac{c^2}{a}=0$.
Results of the applied selection method are presented in Table 1.

Table 1 Selection of integers

c^2	a	$\frac{c^2}{a}=x$	b^2	b^2c^2	$\frac{b^2}{a}\cdot\frac{c^2}{a}$	p^2	$\left(x-\frac{c^2}{a}\right)=0$	$p^2-\frac{b^2}{a}\cdot\frac{c^2}{a}=0$
16	8	2	100	1600	25	25	0	0
36	12	3	64	2304	16	16	0	0
36	9	4	144	5184	64	64	0	0
64	16	7	144	9216	36	36	0	0

We find the values of y, such that $\left(\frac{b^2}{a}-y\right)$ is an integer or b^2-ay can be factored by a.

As a result we get:

$$a=8, b^2=100\Rightarrow\left(\frac{100}{8}-y\right)=k\in Z\Rightarrow y=6,$$

$$a=12, b^2=64\Rightarrow\left(\frac{64}{12}-y\right)=k\in Z\Rightarrow y=9,$$

$$a=9, b^2=144\Rightarrow\left(\frac{144}{9}-y\right)=k\in Z\Rightarrow y=5,$$

$$a=16, b^2=144\Rightarrow\left(\frac{144}{16}-y\right)=k\in Z\Rightarrow y=9.$$

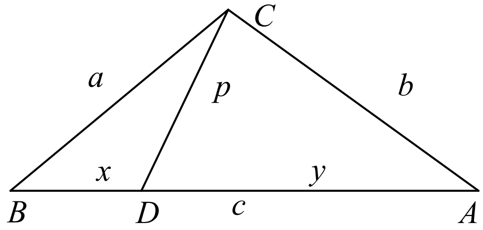


Figure 6

Thus, after calculations we have computed four different integer solutions of the Stewart's equation in a form that corresponds to the equilateral triangles in notations of Fig. 6 (Fig. 7).

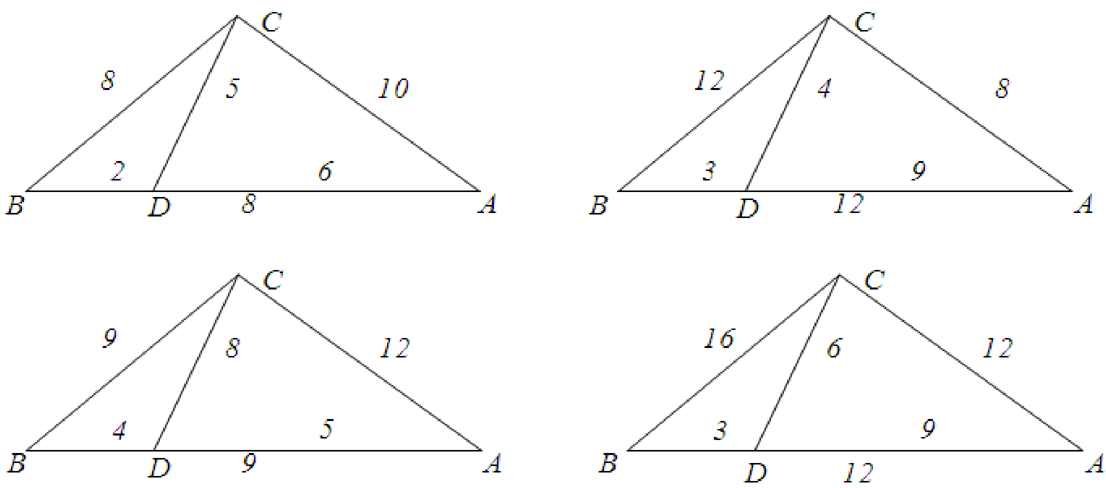


Figure 7

We consider the prospect of further research to investigate algorithms for finding integer solutions of the Stewart's equation, provided an in-depth study of the solutions of first- and second-order Diophantine equations, which have the basis of Fibonacci and Lucas numbers, and an understanding of the Fermat's little theorem and Fermat's last theorem.

CONCLUSIONS

In this paper the Stewart's theorem was considered as one of the classical problems of geometry of a triangle and its application to different problems. We studied the problem of computing integer solutions of Diophantine equations and presented some results of studies of the Stewart's theorem in view of finding its integer solutions, that is related to first- and second-order Diophantine equations. The method for computing integer solutions of Stewart's equations was presented and four different integer solutions for equilateral triangles were computed, using the described method.

From the obtained results we can conclude that application of the Stewart's theorem is an effective tool for solving numbers of geometrical problems. The generalizations of the Stewart's theorem and its applications to computing the length of a segment, that connects the vertex of a triangle with its inner point, can be applied in logistics, management and designing. For example, to choose the right coordinates for set of three stores, which are the vertices of the triangle or to find correct place for Wi-Fi router for a few settlements (for three, for example).

REFERENCES

[1] *Geometry*. Retrieved 2019-08-01 from http://math.cmu.edu/~cargue/arml/archive/13-14/geometry-12-15-13.pdf

[2] Kadubovska, V. M., Kadubovska, O. L., & Kadubovskij, O. A. (2012). *Navkolo Teoremi Styuarta: naslidki, uzagaljnenja ta zastosuvanja* (Навколо теореми Стюарта: наслідки, узагальнення та застосування, in Ukrainian). Retrieved 2019-08-02 from https://www.researchgate.net/publication/277141893_

[3] Keskin, R. (2010). Solutions of some quadratic Diophantine equations. *Comput. Math. Appl.*, Vol. 60, № 8, pp. 2225-2230.

[4] Keskin, R. & Bitim, B. (2011). Fibonacci and Lucas congruences and their applications. *Acta Math. Sin., Engl. Ser.*, Vol. 27, № 4, pp. 725-736.

[5] Keskin, R. & Yosma, Z. (2011). On Fibonacci and Lucas numbers of the form cx^2. *J. Integer Seq.*, Vol. 14, № 9, pp. 9-12.

[6] Keskin, R. & Yosma, Z. (2009). Positive integer solutions of the Diophantine equations $x^2-5F_nxy-5\cdot(-1)^n y^2=\pm 5^r$. *J. Integer Seq.,* Vol. 10, № 5, pp. 22-25.

[7] Kolmogorov, N. A. (1936). Analog teoremy i tozdestva Styuarta (Аналог теоремы и тождества Стюарта, in Russian). *Matematiceskoje prosvescenije*, Part 7, pp. 22-25.

[8] Kuschnir, I. A. (2002). Geometritchni formuli, stcho ne vyjschli do schkiljnich pidrucchnikiv: Dovidnik (Геометричні формули, що не ввійшли до шкільних підручників: Довідник, in Ukrainian). K.: Fakt, 112 p.

[9] Matthews, K. (2002). The Diophantine equation $ax^2+bxy+cy^2=N$. *J. Theor. Nombres Bordx.*, Vol. 14, № 1, pp. 257-270.

[10] Melnichenko, O. P. (2012). Naukova dijaljnist u svitli prognozuvanija, postanovka problemi, aktualjnosti, objekta i predmeta (Наукова діяльність у світлі прогнозування, постановка проблеми, актуальності, об'єкта і предмета, in Ukrainian). Naukovij visnik LNUVMB, Vol. 14, № 3(53), pp. 48-51.

[11] Merzljak, A. G., Polonskij, V. G. & Jakir, M. S. (2009). Geometrija: pidrucchnik dlja 9 klasu schkil z pogliblenim vivtchenjam matematiki (Геометрія: підручник для 9 класу шкіл з поглибленим вивченням математики, in Ukrainian). X.: Gimnazija, 272 p.

[12] Mollin, R. (2006). Quadratic Diophantine equations $x^2 - Dy^2 = c^n$. *Ir. Math. Soc. Bull.*, Vol. 58, pp. 55-68.

[13] Mordell, L. (1969). Diophantine equations. *Pure and Applied Mathematics*. London-New York: Academic Press, Vol. 30.

[14] Novosad, M. V. & Ditchka, I. A. (2009). Tchisla Ljuka (Числа Люка, in Ukrainian). *Naukovij visnik Tcherniveckogo universitety*, Vol. 446, pp. 11- 15.

[15] *Proving Stewarts Theorem*. Retrieved 2019-08-09 from https://math.stackexchange.com/questions/46616/proving-stewarts-theorem

[16] Robertson, J. *Matthews' method for solving* $ax^2 + bxy + cy^2 = N$ (manuscript). Retrieved 2019-08-18 from http://www.jpr2718.org/mtthbin.pdf

[17] *Stewart Theorem*. Retrieved 2019-08-09 from https://www.qc.edu.hk/math/Small%20problems/Stewart%20Theorem.pdf

[18] *Stewarts Theorem*. Retrieved 2019-08-18 from https://brilliant.org/wiki/stewarts-theorem/

[19] *Teorema Styuarta*. Retrieved 2019-08-12 from http://hijos.ru/2015/01/23/sootnoshenie-bretshnajdera-i-teorema-styuarta/

[20] Yong, W. & Bound, J. *Using Stewarts´s Theorem*. Retrieved 2019-08-12 from https://sms.math.nus.edu.sg/smsmedley/Vol-31-Using%20Stewart's%20Theorem %20(Willie%20Yong%20and%20Jim%20Boyd)