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Original Paper

Evaluation of specific integrals by differentiation

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ABSTRACT

Differentiation and integration (anti-differentiation) constitute one of the fundamental techniques used in higher mathematics. These operations are inverse of each other. While differentiation (to the extent of school mathematics) is relatively simple and straightforward, integration, in general, is a much more involving task. There are various classical methods to evaluate elementary integrals, e.g. substitution, integration by parts, partial fraction decomposition or more advanced techniques like the residue theorem, or Cauchy's integral formula. The paper deals with some types of elementary functions whose integrals can be evaluated by intelligent guess and differentiation.

KEYWORDS: higher mathematics, differentiation, integration

JEL CLASSIFICATION: I 20, C20

INTRODUCTION

Integration, i.e. evaluation of the indefinite integral is one of the basic operations in higher mathematics. If we have a continuous function $f(x)$ on (a,b), then there exists a function $F(x)$ having the property that $F'(x)=f(x)$ on (a,b). The function $F(x)$ is called an antiderivative of $f(x)$. In order to find the function $F(x)$, we have to integrate or anti-differentiate the function $f(x)$. So simply speaking, integration is a reverse operation to differentiation. Differentiation is a relatively simple and routine operation, since there are general rules available [1], [2], [4]. On the other hand, its reverse, integration is generally much more intricate and a tedious task. In the paper we focus on some functions whose integrals can be guessed and evaluated by subsequent differentiation and comparison. The idea was discussed by Dawson in [3]. Some functions, when differentiated, do not change qualitatively. These functions are polynomials, exponentials and trigonometric functions such as $\sin ax$, $\cos bx$ and various combinations of them. Differentiation of these functions gives back qualitatively the same functions.

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MATERIAL AND METHODS

In some cases we can guess the form of the antiderivative of the function f. The idea behind is the property:

$$\int f(x)\,dx = something \quad \Rightarrow \quad [something]' = f(x) \tag{1}$$

So if "*something*" differentiated is qualitatively the same as the function being integrated, we can equate both sides of (1), make a comparison and obtain a solution. The usual method that works here is the method of undetermined coefficients. We illustrate the idea with a simple example. Let's consider the integral $\int (x^2+x+1)\,dx$. We know that derivatives of polynomials are polynomials of degree less by one, hence the antiderivative of x^2+x+1 must be a polynomial of degree 3, i.e. of the form Ax^3+Bx^2+Cx+D.

$\int (x^2+x+1)\,dx = Ax^3+Bx^2+Cx+D \;\Rightarrow\; x^2+x+1 = \left[Ax^3+Bx^2+Cx+D\right]'$ and

$x^2+x+1 = 3Ax^2+2Bx+C$ and a simple comparison yields $A=\frac{1}{3}, B=\frac{1}{2}, C=1$, and

$$\int (x^2+x+1)\,dx = \frac{1}{3}x^3+\frac{1}{2}x^2+x+D$$

RESULTS AND DISCUSSION

Now we try to develop this idea a bit more, primarily to integrals of the abovementioned functions. Hereinafter, we denote polynomials of degree *n, m* as $P_n(x), Q_m(x)$etc., respectively and their *k*-th derivatives as $P_{n-k}(x), Q_{m-k}(x)$ etc., respectively.

1. $\int P_n(x)\,e^{ax}\,dx.$

A judicious guess says that the antiderivative of $P_n(x)\,e^{ax}$must have the form $Q_n(x)\,e^{ax}$. If we differentiate the function $P_n(x)\,e^{ax}$, we get (since we consider polynomials in general, we deliberately neglect the negative signs and the factor *a*):

$$\left[P_n(x)\,e^{ax}\right]' = P_{n-1}(x)\,e^{ax} + P_n(x)\,e^{ax}$$

and further

$$\left[P_{n-1}(x)\,e^{ax}\right]' = P_{n-2}(x)\,e^{ax} + P_{n-1}(x)\,e^{ax}$$

$$\left[P_{n-2}(x)\,e^{ax}\right]' = P_{n-3}(x)\,e^{ax} + P_{n-2}(x)\,e^{ax}$$

By combining these equations we obtain

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$P_n(x)e^{ax} = \left[P_n(x)e^{ax}\right]' + \left[P_{n-1}(x)e^{ax}\right]' + \left[P_{n-2}(x)e^{ax}\right]' + P_{n-3}(x)e^{ax}$ or

$P_n(x)e^{ax} = \left[\left(P_n(x) + P_{n-1}(x) + P_{n-2}(x)\right)e^{ax}\right]' + P_{n-3}(x)e^{ax}$

It is obvious that after performing n steps the polynomial $P_n(x)$ reduces to a constant and we will have

$$P_n(x)e^{ax} = \left[\left(\sum_{k=0}^{n} P_{n-k}(x)\right)e^{ax}\right]'$$

$$\int P_n(x)e^{ax}\,dx = \left(\sum_{k=0}^{n} P_{n-k}(x)\right)e^{ax} = Q_n(x)e^{ax}, \text{ where } \sum_{k=0}^{n} P_{n-k}(x) = Q_n(x)$$

Example 1: Evaluate $\int (3x^2 - 2x + 4)e^{2x}\,dx$.

Solution:

$\int (3x^2 - 2x + 4)e^{2x}\,dx = (Ax^2 + Bx + C)e^{2x}$

Now we take the derivatives of both sides

$(3x^2 - 2x + 4)e^{2x} = (2Ax + B)e^{2x} + 2\,(Ax^2 + Bx + C)e^{2x}$

We immediately see that $A = \frac{3}{2}, 2A + 2B = -2, B + 2C = 4 \Rightarrow B = -\frac{5}{2}, C = \frac{13}{4}$, hence

$$\int (3x^2 - 2x + 4)e^{2x}\,dx = \left(\frac{3}{2}x^2 - \frac{5}{2}x + \frac{13}{4}\right)e^{2x} + const$$

2. $\int P_n(x)\sin ax\,dx, \int P_m(x)\cos ax\,dx.$

In like manner as in the previous case (by subsequent differentiation and reduction of the polynomial $P_n(x)$ to a constant) we can derive that

$$\int P_n(x)\sin ax\,dx = Q_n(x)\cos ax + R_{n-1}(x)\sin ax \qquad (2)$$

$$\int P_m(x)\cos ax\,dx = Q_m(x)\sin ax + R_{m-1}(x)\cos ax \qquad (3)$$

Example 2: Evaluate $\int (4x^3 - x + 1)\sin 4x\,dx$.

Solution:

$\int (4x^3 - x + 1)\sin 4x\,dx = (Ax^3 + Bx^2 + Cx + D)\cos 4x + (Ex^2 + Fx + G)\sin 4x$

$$\left(4x^3-x+1\right)\sin 4x=\left(3Ax^2+2Bx+C\right)\cos 4x+4\left(-Ax^3-Bx^2-Cx-D\right)\sin 4x+$$
$$4\left(Ex^2+Fx+G\right)\cos 4x+\left(2Ex+F\right)\sin 4x$$

And we have

$$A=-1,\, B=0,\, -3+4E=0,\, -4C+2E=-1,\, 2B+4F=0,\, -4D+F=1,\, C+4G=0$$

$$\int\left(4x^3-x+1\right)\sin 4x\,dx=\left(-x^3+\frac{5}{8}x-\frac{1}{4}\right)\cos 4x+\left(\frac{3}{4}x^2-\frac{5}{32}\right)\sin 4x+const$$

Adding up (2) and (3) yields

$$\int P_n(x)\sin ax+Q_m(x)\cos ax\,dx=R_n(x)\cos ax+S_{n-1}(x)\sin ax,\ \text{ if }\ n>m$$

$$\int P_n(x)\sin ax+Q_m(x)\cos ax\,dx=R_n(x)\cos ax+S_n(x)\sin ax,\ \text{ if }\ n=m$$

$$\int P_n(x)\sin ax+Q_m(x)\cos ax\,dx=R_{m-1}(x)\cos ax+S_m(x)\sin ax,\ \text{ if }\ n<m$$

Example 3: Evaluate $\int\left(\left(x^2+3x-2\right)\sin 6x+\left(x^3+x^2+4\right)\cos 6x\right)dx$.

Solution:

$$\int\left(\left(x^2+3x-2\right)\sin 6x+\left(x^3+x^2+4\right)\cos 6x\right)dx=\left(Ax^2+Bx+C\right)\cos 6x+$$
$$\left(Dx^3+Ex^2+Fx+G\right)\sin 6x$$

$$\left(x^2+3x-2\right)\sin 6x+\left(x^3+x^2+4\right)\cos 6x=\left(2Ax+B\right)\cos 6x+6\left(-Ax^2-Bx-C\right)\sin 6x+$$
$$6\left(Dx^3+Ex^2+Fx+G\right)\cos 6x+\left(3Dx^2+2Ex+F\right)\sin 6x$$

$$D=\frac{1}{6},\, E=\frac{1}{6},\, 2A+6F=0,\, B+6G=4,\, -6A+3D=1,\, -6B+2E=3,\, -6C+F=-2$$

$$\int\left(\left(x^2+3x-2\right)\sin 6x+\left(x^3+x^2+4\right)\cos 6x\right)dx=$$
$$=\left(\frac{-1}{12}x^2-\frac{4}{9}x+\frac{73}{216}\right)\cos 6x+\left(\frac{1}{6}x^3+\frac{1}{6}x^2+\frac{1}{36}x+\frac{20}{27}\right)\sin 6x+const$$

3. $\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx.$

$$\left[e^{ax}\sin bx\right]'=a\,e^{ax}\sin bx+b\,e^{ax}\cos bx \tag{4}$$

$$\left[e^{ax}\cos bx\right]'=-b\,e^{ax}\sin bx+a\,e^{ax}\cos bx\quad /\cdot\frac{a}{b} \tag{5}$$

Adding up (4) and (5)

$$\left[e^{ax}\sin bx\right]' + \frac{a}{b}\left[e^{ax}\cos bx\right]' = \left(\frac{a^2}{b}+b\right)e^{ax}\cos bx$$ and relabeling the constants gives

$$\int e^{ax}\cos bx\,dx = A\,e^{ax}\sin bx + B\,e^{ax}\cos bx$$

Analogously, solving for $\int e^{ax}\sin bx\,dx$ yields the same general form of solution

$$\int e^{ax}\sin bx\,dx = A\,e^{ax}\sin bx + B\,e^{ax}\cos bx$$

Example 4: Evaluate $\int e^{4x}\sin 5x\,dx$.

Solution:

$$\int e^{4x}\sin 5x\,dx = A\,e^{4x}\sin 5x + B\,e^{4x}\cos 5x$$

$$e^{4x}\sin 5x = 4A\,e^{4x}\sin 5x + 5A\,e^{4x}\cos 5x + 4B\,e^{4x}\cos 5x - 5B\,e^{4x}\sin 5x$$

$$4A-5B=1,\ 5A+4B=0$$

$$\int e^{4x}\sin 5x\,dx = \frac{4}{41}e^{4x}\sin 5x - \frac{5}{41}e^{4x}\cos 5x + const$$

4. $\int P_n(x)e^{ax}\sin bx\,dx, \int P_n(x)e^{ax}\cos bx\,dx.$

This case is the combination of all the previous cases. Pursuing the same idea shows that the general form of a solution in this case is

$$\int P_n(x)e^{ax}\sin bx\,dx = Q_n(x)e^{ax}\sin bx + R_n(x)e^{ax}\cos bx$$

$$\int P_m(x)e^{ax}\cos bx\,dx = Q_m(x)e^{ax}\sin bx + R_m(x)e^{ax}\cos bx$$

Example 5: Evaluate $\int (x+4)e^{2x}\cos 3x\,dx$.

Solution:

$$\int (x+4)e^{2x}\cos 3x\,dx = (A\,x+B)e^{2x}\cos 3x + (C\,x+D)e^{2x}\sin 3x$$

$$\begin{aligned}(x+4)e^{2x}\cos 3x = {}& A\,e^{2x}\cos 3x + 2(Ax+B)e^{2x}\cos 3x - 3(Ax+B)e^{2x}\sin 3x +\\ & C\,e^{2x}\sin 3x + 2(Cx+D)e^{2x}\sin 3x + 3(Cx+D)e^{2x}\cos 3x\end{aligned}$$

After canceling e^{2x} we have

$A+2B+3D=4, 2A+3C=1, -3A+2C=0, -3B+C+2D=0$,

and solving this system yields

$$\int (x+4)e^{2x}\cos 3x\,dx = \left(\frac{2}{13}x+\frac{109}{169}\right)e^{2x}\cos 3x + \left(\frac{3}{13}x+\frac{144}{169}\right)e^{2x}\sin 3x + const$$

CONCLUSIONS

Note that all the above mentioned integrals can be evaluated by means of the "by parts" integration method, which is also a formal justification of the results, but employing this method to solve Example 3 or Example 5 is a pretty formidable task, to say the least. There are other types of functions whose antiderivatives can be found without the "necessity" of integration. We will investigate such functions in the upcoming paper. The use of the presented method is left to the reader in every particular case.

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