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Original Paper

Differentiating under integral sign in Castigliano's theorem

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ABSTRACT

In this contribution we are dealing with application of differentiating of function defined with determined parametrical integral. The mathematical problem was analyzed form point of view of mechanics of materials. The mathematical model of loaded beam was created. Applying the cross section method we defined the exact function of bending moment. Additional properties of cantilever joint were neglected. We showed the derivation of modified Castigliano's theorem via Leibnitz rule of differentiating under integration sign. Appling the modified Castigliano's theorem we got the exact solution of the deflection of the beam. The exact solution of beam deflection was finished in PTC Mathcad Prime software (PTC - Parametric Technology). The numerical integration of the bending moment dataset and defined deflection function was done in program written in Microsoft Visual C# 2010. The Dormand-Prince numerical integration method was used for the numerical integration. Comparing the exact and numerical solution we got the error of numerical integration solution.

KEYWORDS: differentiating methods, beam deflection, parametric integral, numerical integration

JEL CLASSIFICATION: C02

INTRODUCTION

The application of modified Castigliano's theorem is the basic knowledge for the structural and civil engineering. It's have been published a many articles which deals with the application this method. But, unfortunately the some major parts are still missing in the published articles. The goals of this paper are focused on the summarization of the way how it was created and how does it work the modified Castigliano's theorem in the application in mechanical engineering. The application of generalized form of Castigliano's theorem was published by [11]. Of course, the method were replaced by the finite element method (FEM) applied in the continuum mechanic of elastic bodies. But many technical applications are the technical functions which are defined by the integral. These functions must be differentiated manually or numerically. The mathematic procedure including the differentiating the function defined by integral was analyzed by [12]. The method of differentiating the

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functions under integral sign was analyzed by [1, 5, 8, 14]. Many mathematical examples of differentiating under integral sign were published by [13] Explaining the visualization of Leibnitz rule was done by [6]. The deflection of machines parts applying the Castigliano's theorem was realized by [7, 17].

MATERIAL AND METHODS

The least energy method

The energetic method of determination of beam deflection y is defined in [3]. The main idea of the method is the determination of the difference dA of the strain energy function A of more variables where $A = f(F_1,..,F_i,..,F_n; M_1,...,M_i,..,M_n)$ by using the total derivate in the differentiating form as follows:

$$dA = \frac{\partial A}{\partial F_1}dF_1 + ... + \frac{\partial A}{\partial F_i}dF_i + ... + \frac{\partial A}{\partial F_n}dF_n + ... + \frac{\partial A}{\partial M_1}dM_1 + ... + \frac{\partial A}{\partial M_i}dM_i + ... + \frac{\partial A}{\partial M_n}dM_n \quad (1)$$

where $1 \le n \le +\infty$. The energy in deformable body is separated to two parts as described in [3]. The significant and not neglected part of strain energy has the form $dA_i = dF_i.y_i$.

Selection of $i-th$ member from equation (1) and compare them we got $dF_i.y_i = \frac{\partial A}{\partial F_i}dF_i$ and finally we got the equation for solving the deflection $y_i = \frac{\partial A}{\partial F_i}$. The general form of this equation is follows:

$$y = \frac{\partial A}{\partial F}. \quad (2)$$

Differentiating under integral sign

Let is defined the function of two variables by definite integral with boundaries $a(x), b(x)$ of the continuous function where as defined in [9]:

$$I(x,t) = \int_{a(x)}^{b(x)} f(x,t). \quad (3)$$

Differentiating function $I(x,t)$ with using the limits we got:

$$\frac{\partial I(x,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{I(x,t+\Delta t) - I(x,t)}{\Delta t}.$$

After the some equations arrangements published in [2] we got the final form:

$$\frac{\partial I(x,t)}{\partial t} = f[b(x),t].\frac{d}{dt}b(x) - f[a(x),t].\frac{d}{dt}a(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial t}dx. \quad (4)$$

Mathematical model

We were self-created the model of isotropic beam loaded with continuous triangular loading q_2. The self-mass of beam is modelled with uniform loading q_1. The beam section profile is square and its dimensions are in table 1.The model is depicted on the Figure 1. The model properties are in Table 1.

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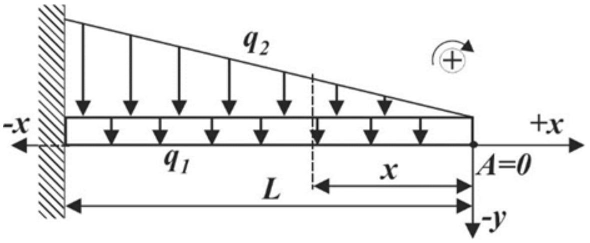

Figure 1 Model of loaded beam

Table1 Beam properties

Parameter	Value	Unit
Beam length L	1	m
Beam profile $a \times a \times t$	0,05x0,05x0,003	m
Beam mass m_B	4,16	kg
Uniform loading q_1	40,81	$N.m^{-1}$
Uniform loading q_2	4081	$N.m^{-1}$
Moment of inertia J_z	2,08492.10^{-7}	m^4
Modulus of elasticity E	2,1.10^{11}	Pa

For any position x of the cross-section the bending moment has the form in equation (5):

$$M_x = -\left(\frac{q_1.x^2}{2} + \frac{q_2.x^3}{6.L} \right) \tag{5}$$

The bending moment of the loaded beam is depicted in the Figure 2. The cantilever joint position of the beam is located in the $L = 1m$ dimension of the beam length (opposite coordinate system).

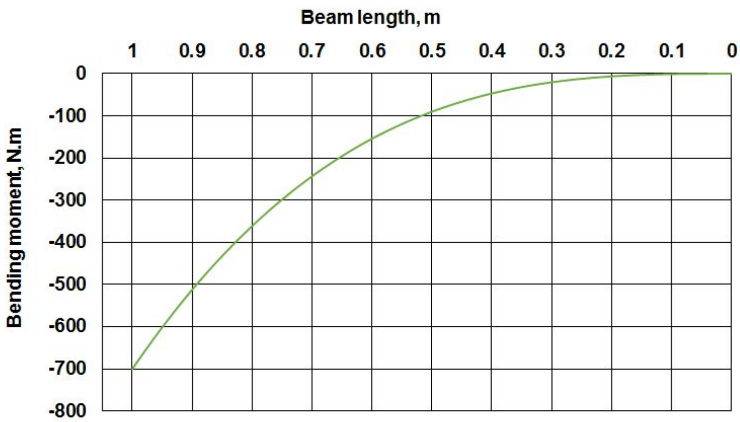

Figure 2 Bending moment function visualization

The exact solution of beam deflection on the end of beam (point A) has the form in equation (6).

$$y_A = -\left(\frac{q_1.L^4}{8.E.J_z} + \frac{q_2.L^4}{30.E.J_z} \right) \tag{6}$$

For modelling the deflection of the beam we rewrite the equation (6) to the form in equation (7).

$$y_{(i)} = -\left(\frac{q_1}{8.E.J_z} + \frac{q_2}{30.E.J_z} \right).x_{(i)}^4 \tag{7}$$

Numerical solution of deflection

Dormand – Prince method is addressed by its authors [4]. They presented this method in the form of Butcher table, where are described coefficients of particular terms in the equations. The equations for Dormand – Prince method are as follows:

$$k_1 = h_i f\left(x_i, y_i\right),\; k_2 = h_i f\left(x_i + \frac{1}{5}h_i, y_i + \frac{1}{5}k_1\right),$$
$$k_3 = h_i f\left(x_i + \frac{3}{10}h_i, y_i + \frac{3}{40}k_1 + \frac{9}{40}k_2\right),$$
$$k_4 = h_i f\left(x_i + \frac{4}{5}h_i, y_i + \frac{44}{45}k_1 - \frac{56}{15}k_2 + \frac{32}{9}k_3\right),$$
$$k_5 = h_i f\left(x_i + \frac{8}{9}h_i, y_i + \frac{19372}{6561}k_1 - \frac{25360}{2187}k_2 + \frac{64448}{6561}k_3 - \frac{212}{729}k_4\right) \qquad (8)$$
$$k_6 = h_i f\left(x_i + h_i, y_i + \frac{9017}{3168}k_1 - \frac{355}{33}k_2 - \frac{46732}{5247}k_3 + \frac{49}{176}k_4 - \frac{5103}{18656}k_5\right)$$
$$k_7 = h_i f\left(x_i + h_i, y_i + \frac{35}{384}k_1 + \frac{500}{1113}k_3 + \frac{125}{192}k_4 - \frac{2187}{6784}k_5 + \frac{11}{84}k_6\right)$$
$$y_{i+1} = y_i + \frac{35}{284}k_1 + \frac{500}{1113}k_3 + \frac{125}{192}k_4 - \frac{2187}{9784}k_5 + \frac{11}{84}k6.$$

The numerical integration algorithm was created in the Microsoft Visual C# 2010 language. The algorithms of solving the differential equations with families of Runga-Kutta methods of higher degrees were published by [15].

RESULTS AND DISCUSSION

We define the strain energy in deformable body under the bending loading and it has the next form: $A = \frac{1}{2E.J}\int_0^L M_x^2 dx$. (9)

Placing the equation (9) to equation (2) we got the form: $y = \frac{\partial}{\partial F}\left(\frac{1}{2E.J}\int_0^L M_x^2 dx\right)$. Applying the equation (4) and setting the boundaries for integral $a(x)=0, b(x)=L$ we got:

$$\frac{\partial I\left(M_x, F\right)}{\partial F} = \frac{1}{2E.J_z}\left\{f[L,F].\frac{d}{dt}(L) - f[0,F].\frac{d}{dt}(0) + \int_0^L \frac{\partial f(M_x^2, F)}{\partial F}dx\right\}. \qquad (10)$$

The next parts of equation (10) are zero: $f[L,F].\frac{d}{dt}(L) = 0$, $f[0,F].\frac{d}{dt}(0) = 0$.

On the next step we got for deflection:

$$y = \frac{1}{2E.J}\int_0^L \frac{\partial M_x^2}{\partial F}dx. \qquad (11)$$

The square of bending moment we should rewrite as follows: $y=\frac{1}{2E.J}\int_0^L \frac{\partial(M_x.M_x)}{\partial F}dx$.

Applying the rule for differentiating the two variables product in general form $(u.v)'=u'v+uv'$, we got: $\frac{\partial(M_x.M_x)}{\partial F}=\frac{\partial M_x}{\partial F}M_x+M_x\frac{\partial M_x}{\partial F}=2\frac{\partial M_x}{\partial F}M_x$.

Returning to back the equation (11) we have the final form of modified Castigliano's theorem:

$$y=\frac{1}{E.J}\int_0^L \frac{\partial M_x}{\partial F}M_x dx. \tag{12}$$

If we want to define the function of bending moment for application the equation (12), we have to insert to the certain point (where we are looking for the value of deflection) the force F equal to zero. The function of bending moment for our model will be the next:

$$M_x=-\left(\frac{q_1.x^2}{2}+\frac{q_2.x^3}{6.L}+F.x\right) \tag{13}$$

Combining the equation (12) and (13) we got:

$$y=\frac{1}{E.J}\int_0^L \left\{\frac{\partial}{\partial F}\left[-\left(\frac{q_1.x^2}{2}+\frac{q_2.x^3}{6.L}+F.x\right)\right]\right\}.\left[-\left(\frac{q_1.x^2}{2}+\frac{q_2.x^3}{6.L}+F.x\right)\right]dx.$$

Result of the partial differential (in {} brackets) is the next: $(-x)$, and in the integral part (in the [] brackets) of the equation holds $F=0$ and finally we got:

$$y=\frac{1}{E.J}\int_0^L x\left(\frac{q_1.x^2}{2}+\frac{q_2.x^3}{6.L}\right)dx. \tag{14}$$

Solving the integral (14) we got the equation (6). Rewriting the equation (6) for displaying the deflection we got the equation (7). Generating the deflection curve with step $\Delta x=0.01m$ we got the curve depicted in the Figure 3.

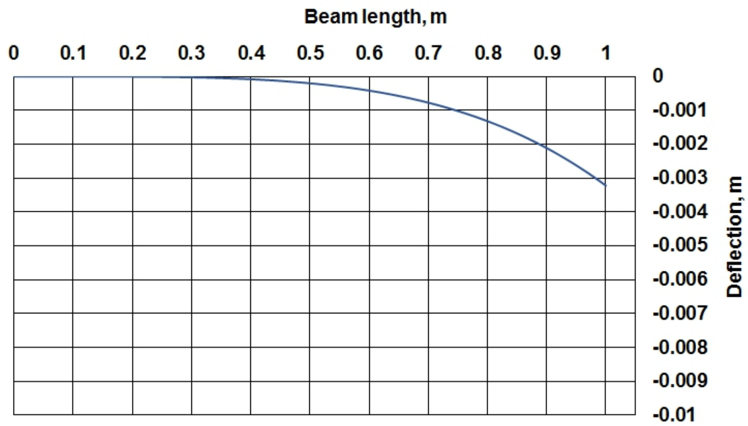

Figure 3 Deflection curve of the loaded beam form exact solution

With the exact solution of the deflection of the beam on the end of beam ($L=1m$) we got the value $y_{ex}=-3,2234783322355.10^{-3}m$. The exact solution was realized in PTC Mathcad Prime software. For numerical solution we rewrite the equation (14) to the next form:

$$y_{(i)} = \frac{1}{E.J}\int_0^L x_{(i)}.M_{x(i)}dx \,. \tag{15}$$

Applying the Dormand-Prince numerical method in equations (8) to solve integral equation (15), we got the data set of beam deflection curve depicted in the Figure 4. The numerical integration step was $h_i = 0{,}01$. The efficiency of the used numerical methods were declared by [10,16].

Figure 4 Deflection curve obtained by numerical integration

Finally we should compare the result of deflection obtained from exact solution with results obtained via numerical integrations. The difference (error) was solved from the equation (16).

$$E_{r(i)} = y_{ex(i)} - y_{num(i)} \,, \tag{16}$$

where: $y_{ex(i)}$ is the deflection dataset solved via exact solution, $y_{num(i)}$ is the deflection dataset solved via numerical integration, where $i\langle 0,n\rangle$. The error function is depicted in the Figure 5. The similar numerical problem was investigated and evaluated by [7].

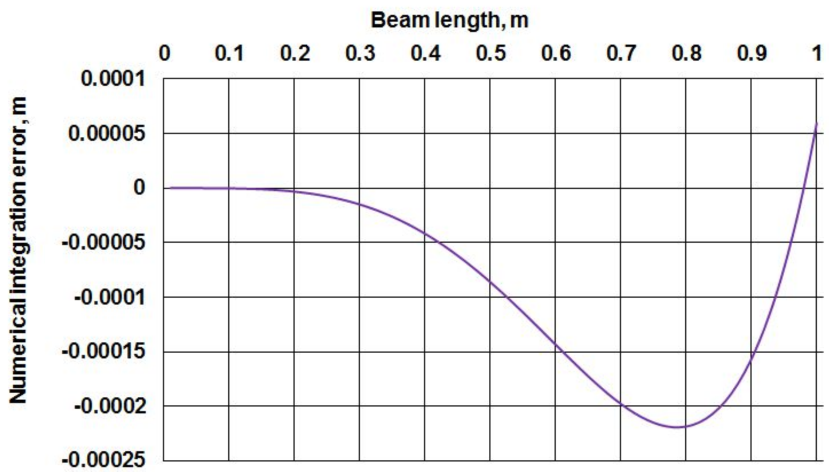

Figure 5 Error of numerical integration

The acceptable errors of numerical integration were discussed in [4].

CONCLUSIONS

In this paper we are dealing with the method of exact and numerical solving of the loaded beam deflection. For the exact solution was chosen the modified Castigliano's theorem. We showed the derivation of the modified Castigliano's theorem through the Leibnitz rule of differentiating under integral sign on defined example. We set up the mathematical model. For the exact solution we solved and visualized the function of bending moment and the beam deflection. For this purpose we used the PTC Mathcad Prime software. For the defined function of deflection we set up the algorithm of numerical integration in Microsoft Visual C# language. Used numerical method was Dormand-Prince method. The numerical integration step was chosen $h_i = 0,01$. From exact solution we got the beam deflection on the length $L = 1m$, $y_{ex} = -3,2234783322355.10^{-3} m$ and from the numerical integration $y_{num} = -3.15589088264663.10^{-3} m$. The error in the point $L = 1m$ is $E_r = 6,75874495887.10^{-5} m$.

From the realized analysis we should conclude that the presented method of differentiating under integral sign has a significant role in problems taught in mechanics of materials. Very pure explanation of the Leibnitz rule in the literature is now fixed. The presented methods are utilizable in simple engineering design process or in teaching process in mechanics of materials subject or applied mathematics. The applied numerical integration method has an acceptable accuracy $E_{r\%} = 6,75874495887.10^{-3}\%$. The percentage error was solved follows: $E_{r\%} = \left(|y_{ex}| - |y_{num}|\right)/100$.

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