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Original Paper

The sum of the series of reciprocals of the cubic polynomials with one zero and double non-zero integer root

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ABSTRACT

This contribution is a follow-up to author's previous published papers and deals with the sum of the series of reciprocals of the cubic polynomials with one zero and double non-zero integer root. We derive the formula for the sum of these series and verify it by some examples using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers who are teaching the topic Infinite series or as a subject matter for work with talented students.

KEYWORDS: sum of the series, telescoping series, generalized harmonic number, digamma function, Euler's constant, computer algebra system Maple

JEL CLASSIFICATION: I30

INTRODUCTION

This scientific paper is following published contributions [2], [3] and [4], and deals with the sum of the special series. Let us recall some basic terms. The series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

converges to a limit s if and only if the sequence of partial sums $\{s_n\} = \{a_1 + a_2 + \cdots + a_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$. The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. The nth *harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

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where $H(0)$ is being defined as 0. The *generalized harmonic number* of order n in power r is the sum

$$H(n,r) = \sum_{k=1}^{n} \frac{1}{k^r},$$

where $H(n,1) = H(n)$ are harmonic numbers. Every generalized harmonic number of order n in power 2 can be written as a function of harmonic numbers using formula (see [5])

$$H(n,2) = \sum_{k=1}^{n-1} \frac{H(k)}{k(k+1)} + \frac{H(n)}{n}.$$

Basic and as well interesting information about harmonic numbers can be found in [1].

The *telescoping series* is any series where nearly every term cancels with a preceding or following term, so its partial sums eventually only have a fixed number of terms after cancellation. For example, the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k-3)},$$

where obviously the summation index $k \neq 3$, has the general kth term, after partial fraction decomposition, in a form

$$a_k = \frac{1}{k(k-3)} = \frac{1}{3}\left(\frac{1}{k-3} - \frac{1}{k}\right).$$

After that we arrange the terms of the nth partial sum $s_n = a_1 + a_2 + a_4 + \cdots + a_n$ in a form where can be seen what is cancelling. Then we find the limit $\lim_{n\to\infty} s_n$ of the sequence of the partial sums s_n in order to find the sum s of the infinite telescoping series. In our case we get

$$s_n = \frac{1}{3}\left[\left(-\frac{1}{2} - \frac{1}{1}\right) + \left(-\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right.$$
$$\left.\cdots + \left(\frac{1}{n-6} - \frac{1}{n-3}\right) + \left(\frac{1}{n-5} - \frac{1}{n-2}\right)\right] + \left(\frac{1}{n-4} - \frac{1}{n-1}\right) + \left(\frac{1}{n-3} - \frac{1}{n}\right)\Bigg] =$$
$$= \frac{1}{3}\left(-3 + 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n-2} - \frac{1}{n-1} - \frac{1}{n}\right).$$

So we have

$$s = \lim_{n\to\infty} \frac{1}{3}\left[-3 + H(3) - \frac{1}{n-2} - \frac{1}{n-1} - \frac{1}{n}\right] = \frac{1}{3}\left(-3 + \frac{11}{6}\right) = \frac{-7}{18} = -0{,}3\overline{8}.$$

THE SUM OF THE SERIES OF RECIPROCALS OF THE CUBIC POLYNOMIALS WITH ONE ZERO AND DOUBLE NON-ZERO INTEGER ROOT

Let us consider the series of reciprocals of the normalized cubic polynomials with one zero and double non-zero integer root a, i.e. the series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{k(k-a)^2}, \tag{1}$$

and let us determine its sum $s(0, a_2)$. We differentiate two cases – a case of negative integer root a and a case of positive integer root a.

i) First, let us assume the case of a negative integer root a and denote the sum of the series (1) by $s(0, a_2^-)$. Then, after partial fraction decomposition, we get the general kth term in a form

$$a_k = \frac{1}{k(k-a)^2} = \frac{A}{k} + \frac{B}{(k-a)^2} + \frac{C}{k-a},$$

so $1 = A(k-a)^2 + Bk + Ck(k-a)$. For k = 0 we have $A = 1/a^2$, for $k = a$ is $B = 1/a$ and by comparing coefficients in second powers is $0 = A + C$, whence $C = -1/a^2$, so

$$a_k = \frac{1}{k(k-a)^2} = \frac{1}{a^2}\cdot\frac{1}{k} + \frac{1}{a}\cdot\frac{1}{(k-a)^2} - \frac{1}{a^2}\cdot\frac{1}{k-a} = \frac{1}{a^2}\left(\frac{1}{k} - \frac{1}{k-a}\right) + \frac{1}{a}\cdot\frac{1}{(k-a)^2}.$$

Example 1 Using nth partial sum calculate the sum $s(0, -3_2)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)^2}.$$

Because

$$a_k = \frac{1}{k(k+3)^2} = \frac{1}{9}\left(\frac{1}{k} - \frac{1}{k+3}\right) - \frac{1}{3}\cdot\frac{1}{(k+3)^2},$$

then the nth partial sum $s_n(0, -3_2) = a_1 + a_2 + \cdots + a_n$ is

$$s_n(0,-3_2) = -\frac{1}{3}\left[\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{(n+3)^2}\right] + \frac{1}{9}\left[\left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \right.$$

$$\left. + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n+1}\right) + \left(\frac{1}{n-1} - \frac{1}{n+2}\right) + \left(\frac{1}{n} - \frac{1}{n+3}\right)\right] =$$

$$= -\frac{1}{3}\left[\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{(n+3)^2}\right] + \frac{1}{9}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right).$$

Considering the facts that for any positive integer c is $\lim_{n\to\infty}(1/c) = 1/c$ and $\lim_{n\to\infty}[1/(n+c)] = 0$ and that

$$\lim_{n\to\infty} H(n,2) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

whence

$$\lim_{n\to\infty}\left[\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{(n+3)^2}\right] = \lim_{n\to\infty}[H(n+3,2) - H(3,2)] = \frac{\pi^2}{6} - \frac{49}{36},$$

we get

$$s(0,-3_2) = \lim_{n\to\infty} s_n(0,-3_2) = -\frac{1}{3}\left[\frac{\pi^2}{6} - H(3,2)\right] + \frac{H(3)}{9} = \frac{49 - 6\pi^2}{108} + \frac{11}{54} = \frac{71 - 6\pi^2}{108}$$
$$\doteq 0.1091.$$

Clearly, for arbitrary negative integer a is $A = -a$ positive integer, so $k - a = k + A$ are also positive integers for all $k = 1, 2, \ldots, n$. After cancellation all of the inner terms we get the nth partial sum $s_n = a_1 + a_2 + \cdots + a_n$ in a form

$$s_n(0, a_2^-) = \sum_{k=1}^{n}\left[\frac{1}{A^2}\left(\frac{1}{k} - \frac{1}{k+A}\right) - \frac{1}{A}\cdot\frac{1}{(k+A)^2}\right] = -\frac{1}{A}\left[\frac{1}{(1+A)^2} + \frac{1}{(2+A)^2} + \cdots\right.$$

$$\cdots+\frac{1}{(n+A)^2}\Bigg]+\frac{1}{A^2}\Bigg[\left(\frac{1}{1}-\frac{1}{1+A}\right)+\left(\frac{1}{2}-\frac{1}{2+A}\right)+\cdots+\left(\frac{1}{A}-\frac{1}{2A}\right)+$$
$$+\left(\frac{1}{1+A}-\frac{1}{1+2A}\right)+\cdots+\left(\frac{1}{n-A}-\frac{1}{n}\right)+\left(\frac{1}{n-A+1}-\frac{1}{n+1}\right)+\cdots$$
$$\cdots+\left(\frac{1}{n-1}-\frac{1}{n+A-1}\right)+\left(\frac{1}{n}-\frac{1}{n+A}\right)\Bigg]=$$
$$=-\frac{1}{A}\left[\frac{\pi^2}{6}-H(A,2)\right]+\frac{1}{A^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{A}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{n+A}\right).$$

Considering the three facts above we have

$$s(0,a_2^-)=\lim_{n\to\infty}s_n(0,a_2^-)$$
$$=-\frac{1}{A}\lim_{n\to\infty}\left[\frac{\pi^2}{6}-H(A,2)\right]+\frac{1}{A^2}\lim_{n\to\infty}\left[H(A)-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{n+A}\right].$$

Altogether, we derived this statement:

Theorem 1 The series

$$\sum_{k=1}^{\infty}\frac{1}{k(k-a)^2}$$

has for a negative integer a the sum in the form

$$s(0,a_2^-)=\frac{\pi^2}{6a}-\frac{H(-a,2)}{a}+\frac{H(-a)}{a^2}, \tag{2}$$

where $H(n)$ is the harmonic number and $H(n,2)$ is the generalized harmonic number.

Let us note, for example, that $s(0,-1_2)=2-\pi^2/6$ and $s(0,-2_2)=1-\pi^2/12$.

ii) Now, let us assume the case of a positive integer root a and denote the sum of the series (1) by $s(0,a_2^+)$. Then, after partial fraction decomposition, we also get, as in the case **i)**, that

$$a_k=\frac{1}{a^2}\left(\frac{1}{k}-\frac{1}{k-a}\right)+\frac{1}{a}\cdot\frac{1}{(k-a)^2}.$$

Let us note that the summation index k must be different from a to avoid division by zero.

Example 2 Using nth partial sum calculate the sum $s(0,3_2)$ of the series

$$\sum_{\substack{k=1\\k\neq 3}}^{\infty}\frac{1}{k(k-3)^2}.$$

Because

$$a_k=\frac{1}{k(k-3)^2}=\frac{1}{9}\left(\frac{1}{k}-\frac{1}{k-3}\right)+\frac{1}{3}\cdot\frac{1}{(k-3)^2},$$

then the nth partial sum $s_n(0,3_2)=a_1+a_2+a_4+\cdots+a_n$ is

$$s_n(0,3_2)=\frac{1}{3}\left[\frac{1}{2^2}+\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(n-3)^2}\right]+$$

$$+\frac{1}{9}\left[\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{1}\right)+\left(\frac{1}{4}-\frac{1}{1}\right)+\left(\frac{1}{5}-\frac{1}{2}\right)+\cdots\right.$$

$$\left.\cdots+\left(\frac{1}{n-3}-\frac{1}{n-6}\right)+\left(\frac{1}{n-2}-\frac{1}{n-5}\right)+\left(\frac{1}{n-1}-\frac{1}{n-4}\right)+\left(\frac{1}{n}-\frac{1}{n-3}\right)\right]=$$

$$=\frac{1}{3}\left[\frac{5}{4}+\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(n-3)^2}\right]+\frac{1}{9}\left(3-\frac{1}{1}-\frac{1}{2}-\frac{1}{3}+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right).$$

Analogously, as in the case **i)**, we get

$$s(0,3_2)=\lim_{n\to\infty}s_n(0,3_2)=\frac{1}{3}\left(\frac{5}{4}+\frac{\pi^2}{6}\right)+\frac{3-H(3)}{9}=\frac{15+2\pi^2}{36}+\frac{7}{54}=\frac{6\pi^2+59}{108}$$
$$\doteq 1.0946.$$

For arbitrary positive integer a, after cancellation all of the inner terms, we get the nth partial sum $s_n=a_1+a_2+\cdots+a_{a-1}+a_{a+1}+a_{a+2}+\cdots+a_n$ in a form

$$s_n(0,a_2^+)=\sum_{\substack{k=1\\k\neq a}}^{n}\left[\frac{1}{a^2}\left(\frac{1}{k}-\frac{1}{k-a}\right)+\frac{1}{a}\cdot\frac{1}{(k-a)^2}\right]=\frac{1}{a}\left[\frac{1}{(a-1)^2}+\frac{1}{(a-2)^2}+\cdots\right.$$

$$\left.\cdots+\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(n-a)^2}\right]+\frac{1}{a^2}\left[\left(\frac{1}{1}+\frac{1}{a-1}\right)+\left(\frac{1}{2}+\frac{1}{a-2}\right)+\cdots+\left(\frac{1}{a-1}+\frac{1}{1}\right)\right.$$

$$+\left(\frac{1}{a+1}-\frac{1}{1}\right)+\left(\frac{1}{a+2}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{2a-1}-\frac{1}{a-1}\right)+\left(\frac{1}{2a}-\frac{1}{a-2}\right)+\cdots$$

$$\left.+\left(\frac{1}{n-1}-\frac{1}{n-a-1}\right)+\left(\frac{1}{n}-\frac{1}{n-a}\right)\right]=$$

$$=\frac{1}{a}[H(a-1,2)+H(n-a,2)]+\frac{1}{a^2}\left[H(n)-\frac{1}{a}-H(n-a)+H(a-1)\right].$$

Therefore we have

$$s(0,a_2^+)=\lim_{n\to\infty}s_n(0,a_2^+)=\frac{1}{a}\left[H(a-1,2)+\frac{\pi^2}{6}\right]+\frac{1}{a^2}\left[H(a-1)-\frac{1}{a}\right].$$

Theorem 2 The series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{k(k-a)^2}$$

has for a positive integer a the sum in the form

$$s(0,a_2^+)=\frac{\pi^2}{6a}+\frac{H(a-1,2)}{a}+\frac{H(a-1)}{a^2}-\frac{1}{a^3}, \qquad (3)$$

where $H(n)$ is the harmonic number and $H(n,2)$ is the generalized harmonic number.

For example, $s(0,1_2)=\pi^2/6-1$, $s(0,2_2)=\pi^2/12+1/2+1/4-1/8=\pi^2/12+5/8$. Because formulas (2) and (3) are very similar, they can be obviously expressed by the only common formula given in the following statement.

Corollary 1 The series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{k(k-a)^2}$$

has for a non-zero integer a the sum in the form

$$s(0,a_2) = \frac{\pi^2}{6a} + \frac{H\left(|a| - \frac{\operatorname{sgn}(a)+1}{2}, 2\right)}{|a|} + \frac{H\left(|a| - \frac{\operatorname{sgn}(a)+1}{2}\right)}{a^2} - \frac{\operatorname{sgn}(a)+1}{2a^3}, \qquad (4)$$

where $H(n)$ is the harmonic number and $H(n,2)$ is the generalized harmonic number.

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(0,a_2)$ for $a = -10, -9, \ldots, -1$ and for $a = 1, 2, \ldots, 10$. We use on the one hand an approximative direct evaluation of the sum

$$s(0,a_2,t) = \sum_{\substack{k=1 \\ k\neq a}}^{t} \frac{1}{k(k-a)^2},$$

where $t = 500000$, using the basic programming language of the computer algebra system Maple 16, and on the other hand the formula (4) for evaluation the sum $s(0,a_2)$. We compare 20 pairs of these ways obtained sums $s(0,a_2,500000)$ and $s(0,a_2)$ to verify the formula (4). We use following simple procedure `rp30aa` and following two `for` statements:

```
> rp30aa:=proc(a,t)
      local k,s0a2t,s0a2; s0a2t:=0;
      for k from 1 to t do
          if k<>a then s0a2t:=s0a2t+1/(k*(k-a)*(k-a))
                  else s0a2t:=s0a2t+0;
          end if;
      end do;
      s0a2:=Pi*Pi/(6*a)+harmonic(abs(a)-(signum(a)+1)/2,2)/abs(a) +harmonic(abs(a)-
      (signum(a)+1)/2)/(a*a)-(signum(a)+1)/(2*a*a*a);
      print("s(",a,")=",evalf[10](s0a2));
      print("s(",a,t,"=",evalf[10](s0a2t));
      print("diff=",evalf[10](abs(s0a2t-s0a2)));
  end proc:

> for i from -10 to -1 do
      rp30aa(i,500000);
  end do;

> for i from 1 to 10 do
      rp30aa(i,500000);
  end do;
```

Table 1 The approximate values of the sums $s(0,a_2)$

a	-10	-9	-8	-7	-6
$s(0,a_2)$	0.0197730489	0.0232403856	0.0338958782	0.0338958782	0.0424646925
a	-5	-4	-3	-2	-1
$s(0,a_2)$	0.0550687422	0.0748775942	0.1090960516	0.1775329664	0.3550659331
a	1	2	3	4	5
$s(0,a_2)$	0.6449340668	1.4474670340	1.0946076520	0.8504696278	0.6890423689
a	6	7	8	9	10
$s(0,a_2)$	0.5768871594	0.4951306848	0.4331516578	0.3846660630	0.3457598625

Source: own modelling in Maple 16

The approximate values of the sums $s(0, a_2)$, for $a = -10, -9, \dots, -1, 1, 2, \dots 9, 10$, obtained by means of the formula (4) are written in Table 1; values of the sums $s(0, a_2)$ are rounded to 10 decimals.

Computation of 20 pairs of the sums $s(0, a_2)$ and $s(0, a_2, 500000)$ took 10 hours and 35 minutes. The absolute errors, i.e. the differences $|s(0, a_2) - s(0, a_2, 500000)|$, are all only between $2 \cdot 10^{-10}$ and 10^{-9}.

CONCLUSIONS

We dealt with the sum of the series of reciprocals of the cubic polynomials with one zero and double non-zero integer root a, i.e. with the series

$$\sum_{\substack{k=1 \\ k \neq a}}^{\infty} \frac{1}{k(k-a)^2}.$$

We derived that the sum of this series is given by formula

$$s(0, a_2) = \frac{\pi^2}{6a} + \frac{H(|a| - \sigma(a), 2)}{|a|} + \frac{H(|a| - \sigma(a))}{a^2} - \frac{\sigma(a)}{a^3},$$

where $\sigma(a) = [\mathrm{sgn}(a) + 1]/2$, $H(n)$ is the harmonic number and $H(n, 2)$ is the generalized harmonic number. We verified this main result by computing 20 sums by using the computer algebra system Maple 16. The series above so belong to special types of the series, such as geometric and telescoping ones, which sums are given analytically by means of a formula.

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