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Original Paper

The sum of the series of reciprocals of the cubic polynomials with three different positive integer roots

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ABSTRACT

In this contribution we deal with the sum of the series of reciprocals of the cubic polynomials with different positive integer roots. We derive the formula for the sum of these series and verify it by some examples using the basic programming language of the computer algebra system Maple 16. This paper can be an inspiration for teachers who are teaching the topic Infinite series or as a subject matter for work with talented students.

KEYWORDS: sum of the series, telescoping series, harmonic number, digamma function, Euler's constant, computer algebra system Maple

JEL CLASSIFICATION: I30

INTRODUCTION

This contribution is a follow-up to author's papers [2] and [3]. We deal with the sum of the series of reciprocals of the cubic polynomials with different positive integer roots.

Let us recall some basic terms. The series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

converges to a limit s if and only if the sequence of partial sums $\{s_n\} = \{a_1 + a_2 + \cdots + a_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$. The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the square numbers (the *Basel problem*) is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \doteq 1.644934.$$

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The n −th *harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

H_0 being defined as 0. Basic information about harmonic numbers can be found e.g. in the web-site [4] or in [1].

The *telescoping series* is any series where nearly every term cancels with a preceding or following term, so its partial sums eventually only have a fixed number of terms after cancellation. For example, the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)(k-2)(k-3)},$$

where obviously the summation index $k \neq 1,2,3$, i.e. $k = 4, 5, \ldots$, has the general k −th term, after partial fraction decomposition, in a form

$$a_k = \frac{1}{(k-1)(k-2)(k-3)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{2}{k-2} + \frac{1}{k-3}\right).$$

After that we arrange the terms of the n −th partial sum $s_n = a_4 + a_5 + \cdots + a_n$ in a form where can be seen what is cancelling. Then we find the limit $\lim_{n\to\infty} s_n$ of the sequence of the partial sums s_n in order to find the sum s of the infinite telescoping series. In our case we get

$$s_n = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{2}{2} + \frac{1}{1}\right) + \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{2}{4} + \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-3} - \frac{2}{n-4} + \frac{1}{n-5}\right) + \right.$$
$$\left. + \left(\frac{1}{n-2} - \frac{2}{n-3} + \frac{1}{n-4}\right) + \left(\frac{1}{n-1} - \frac{2}{n-2} + \frac{1}{n-3}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{2} - \frac{1}{n-2} + \frac{1}{n-1}\right).$$

So we have

$$s = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2} - \frac{1}{n-2} + \frac{1}{n-1}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

THE SUM OF THE SERIES OF RECIPROCALS OF THE CUBIC POLYNOMIALS WITH THREE DIFFERENT POSITIVE INTEGER ROOTS

Let us consider the series of reciprocals of the normalized cubic polynomials with three different positive integer roots $a < b < c$, i.e. the series

$$\sum_{\substack{k=1 \\ k\neq a,b,c}}^{\infty} \frac{1}{(k-a)(k-b)(k-c)}, \tag{1}$$

and let us determine its sum $s(a,b,c)$. This series can be split into four parts:

$$s(a,b,c) = \sum_{k=1}^{a-1} \frac{1}{(k-a)(k-b)(k-c)} + \sum_{k=a+1}^{b-1} \frac{1}{(k-a)(k-b)(k-c)} +$$
$$+ \sum_{k=b+1}^{c-1} \frac{1}{(k-a)(k-b)(k-c)} + \sum_{k=c+1}^{\infty} \frac{1}{(k-a)(k-b)(k-c)}. \tag{2}$$

The symbol of vertical bar hereinafter used in some of three positions before three letters means that corresponding finite sum in the relation (2) is omitted. For example a notation $s(|a|b,c)$ denotes the case, where $a=1$, $b=2$ and $c-1\geq b+1=3$, i.e. the sum $s(|1|2,c)$.

In total, we differentiate $2^3=8$ following possible cases of the sums: $s(|1|2|3)$, $s(|1|2,c)$, $s(|1,b|b+1)$, $s(a|a+1|a+2)$, $s(a,b|b+1)$, $s(a|a+1,c)$, $s(|1,b,c)$, $s(a,b,c)$.
We focus on the last and most general case of the relation (2), where $0<a<b<c$, $a\geq 2$, $b-1\geq a+1$, $c-1\geq b+1$, and determine the sum $s(a,b,c)$ using the equality

$$\frac{1}{(k-a)(k-b)(k-c)}=\frac{1}{A(k-a)}-\frac{1}{B(k-b)}+\frac{1}{C(k-c)},$$

where $A=(b-a)(c-a)$, $B=(b-a)(c-b)$, $C=(c-a)(c-b)$.
Because

$$\sum_{k=1}^{a-1}\frac{1}{k-a}=\frac{1}{1-a}+\frac{1}{2-a}+\cdots-\frac{1}{2}-\frac{1}{1}=-\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{a-2}+\frac{1}{a-1}\right)=-H_{a-1},$$

$$\sum_{k=1}^{a-1}\frac{1}{k-b}=\frac{1}{1-b}+\frac{1}{2-b}+\cdots+\frac{1}{a-b-2}+\frac{1}{a-b-1}=H_{b-a}-H_{b-1},$$

$$\sum_{k=1}^{a-1}\frac{1}{k-c}=\frac{1}{1-c}+\frac{1}{2-c}+\cdots+\frac{1}{a-c-2}+\frac{1}{a-c-1}=H_{c-a}-H_{c-1},$$

then the sum $s'(a,b,c)$ of the first finite part of the series (2) is

$$s'(a,b,c)=\frac{-H_{a-1}}{A}-\frac{H_{b-a}-H_{b-1}}{B}+\frac{H_{c-a}-H_{c-1}}{C}.$$

Now, let us determine the sum $s''(a,b,c)$ of the second finite part of the series (2). Because

$$\sum_{k=a+1}^{b-1}\frac{1}{k-a}=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{b-a-2}+\frac{1}{b-a-1}=H_{b-a-1},$$

$$\sum_{k=a+1}^{b-1}\frac{1}{k-b}=\frac{1}{a-b+1}+\frac{1}{a-b+2}+\cdots-\frac{1}{2}-\frac{1}{1}=-H_{b-a-1},$$

$$\sum_{k=a+1}^{b-1}\frac{1}{k-c}=\frac{1}{a-c+1}+\frac{1}{a-c+2}+\cdots+\frac{1}{b-c-2}+\frac{1}{b-c-1}=H_{c-b}-H_{c-a-1},$$

then the sum $s''(a,b,c)$ of the second finite part of the series (2) is

$$s''(a,b,c)=\frac{H_{b-a-1}}{A}+\frac{H_{b-a-1}}{B}+\frac{H_{c-b}-H_{c-a-1}}{C}=\left(\frac{1}{A}+\frac{1}{B}\right)H_{b-a-1}+\frac{H_{c-b}-H_{c-a-1}}{C}.$$

Now, we determine the sum $s'''(a,b,c)$ of the third finite part of the series (2). Because

$$\sum_{k=b+1}^{c-1}\frac{1}{k-a}=\frac{1}{b-a+1}+\frac{1}{b-a+2}+\cdots+\frac{1}{c-a-2}+\frac{1}{c-a-1}=H_{c-a-1}-H_{b-a},$$

$$\sum_{k=b+1}^{c-1} \frac{1}{k-b} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{c-b-2} + \frac{1}{c-b-1} = H_{c-b-1},$$

$$\sum_{k=b+1}^{c-1} \frac{1}{k-c} = \frac{1}{b-c+1} + \frac{1}{b-c+2} + \cdots - \frac{1}{2} - \frac{1}{1} = -H_{c-b-1},$$

then the sum $s'''(a,b,c)$ of the third finite part of the series (2) is

$$s'''(a,b,c) = \frac{H_{c-a-1} - H_{b-a}}{A} - \frac{H_{c-b-1}}{B} - \frac{H_{c-b-1}}{C} = \frac{H_{c-a-1} - H_{b-a}}{A} - \left(\frac{1}{B} + \frac{1}{C}\right) H_{c-b-1}.$$

Finally, let us express the partial sum $s_n(a,b,c)$ of the infinite part of the series (2). We have

$$s_n(a,b,c) = \sum_{k=c+1}^{n} \left[\frac{1}{(a^2 - ab - ac + bc)(k-a)} + \frac{1}{(b^2 - ab + ac - bc)(k-b)} + \frac{1}{(c^2 + ab - ac - bc)(k-c)}\right].$$

By means of the computer algebra system Maple 16 we get the following worksheet:

```
> sum((1/(((b-a)^2+(b-a)*(c-b))*(k-a))-1/((b-a)*(c-b)*(k-b))
+1/(((b-a)*(c-b)+(c-b)^2)*(k-c))),k=c+1..n);
```

$$-\frac{\Psi(n+1-b)}{(-c+b)(-b+a)} + \frac{\Psi(n+1-a)}{(a-c)(-b+a)} + \frac{\Psi(n+1-c)}{(-c+b)(a-c)} + \frac{\Psi(c+1-b)}{(-c+b)(-b+a)} - \frac{\Psi(c+1-a)}{(a-c)(-b+a)} + \frac{\gamma}{(-c+b)(a-c)}$$

```
> limit(Psi(n+1-a)/((a-b)*(a-c))-Psi(n+1-b)/((a-b)*(b-c))+Psi(n+1-
c)/((a-c)*(b-c)),n=infinity);
```

$$0$$

```
> simplify(Psi(c+1-b)/((a-b)*(b-c))-Psi(c+1-a)/((a-b)*(a-c))
+gamma/((a-c)*(b-c)));
```

$$\frac{\Psi(c+1-b)a - \Psi(c+1-b)c + \Psi(c+1-a)c - \Psi(c+1-a)b - \gamma b + a\gamma}{(-b+a)(a-c)(-c+b)}$$

Let us note that $\Psi(x)$ is the *digamma function*, $\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\Gamma(x)) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x)}{\Gamma(x)}$, where $\Gamma(x)$ is the *Gamma function* and that *Euler's constant* or *Euler-Mascheroni constant*

$$\gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \ln(n)\right) \doteq 0.5772156649 \dots .$$

Altogether, for $0 < a < b < c$, $a \geq 2$, $b - 1 \geq a + 1$ and $c - 1 \geq b + 1$, we get the sum $s(a,b,c)$ of the series (2) in the form

$$s(a,b,c) = s'(a,b,c) + s''(a,b,c) + s'''(a,b,c) + \lim_{n\to\infty} s_n(a,b,c) = \frac{-H_{a-1}}{A}$$
$$-\frac{H_{b-a} - H_{b-1}}{B} + \frac{H_{c-a} - H_{c-1}}{C} + \left(\frac{1}{A} + \frac{1}{B}\right) H_{b-a-1} + \frac{H_{c-b} - H_{c-a-1}}{C} + \frac{H_{c-a-1} - H_{b-a}}{A}$$
$$+ \frac{(a-c)\Psi(c+1-b) + (c-b)\Psi(c+1-a) + (a-b)\gamma}{(-b+a)(a-c)(-c+b)},$$

i.e.

$$s(a,b,c) = \left(\frac{1}{A} + \frac{1}{B}\right) H_{b-a-1} + \left(\frac{1}{A} - \frac{1}{C}\right) H_{c-a-1} - \left(\frac{1}{B} + \frac{1}{C}\right) H_{c-b-1} - \left(\frac{1}{A} + \frac{1}{B}\right) H_{b-a}$$
$$-\frac{H_{a-1}}{A} + \frac{H_{b-1}}{B} - \frac{H_{c-1}}{C} + \frac{H_{c-a} + H_{c-b}}{C}$$
$$+ \frac{(a-c)\Psi(c+1-b) + (c-b)\Psi(c+1-a) + (a-b)\gamma}{(a-b)(a-c)(b-c)}.$$

Because

$$\frac{1}{A} + \frac{1}{B} = \frac{a+b-2c}{(a-c)(ab+bc-ac-b^2)}, \qquad \frac{1}{B} + \frac{1}{C} = \frac{2a-b-c}{(a-b)(ab-ac-bc+c^2)},$$
$$\frac{1}{A} - \frac{1}{C} = \frac{-a+2b-c}{(a-b)(ab-ac-bc+c^2)},$$

we derived this statement:

Theorem 1. The series

$$\sum_{\substack{k=1 \\ k\neq a,b,c}}^{\infty} \frac{1}{(k-a)(k-b)(k-c)},$$

where $a \geq 2$, $b-1 \geq a+1$ and $c-1 \geq b+1$ are positive integers, has the sum

$$s(a,b,c) = \frac{(a+b-2c)(H_{b-a-1} - H_{b-a})}{(a-c)(ab+bc-ac-b^2)} + \frac{(-a+2b-c)H_{c-a-1} - (2a-b-c)H_{c-b-1}}{(a-b)(ab-ac-bc+c^2)}$$
$$+ \frac{H_{c-a} + H_{c-b} - H_{c-1}}{ab-ac-bc+c^2} - \frac{H_{a-1}}{a^2-ab-ac+bc} + \frac{H_{b-1}}{ab+bc-ac-b^2} \tag{3}$$
$$+ \frac{(a-c)\Psi(c+1-b) + (c-b)\Psi(c+1-a) + (a-b)\gamma}{(a-b)(a-c)(b-c)},$$

where H_n is the $n-$th harmonic number, $\Psi(n)$ is digamma function, and γ is Euler's constant.

Remark 1. It can be shown that the first seven cases of the sums $s(|1|2|3)$, $s(|1|2,c)$, $s(|1,b|b+1)$, $s(a|a+1|a+2)$, $s(a,b|b+1)$, $s(a|a+1,c)$, $s(|1,b,c)$ are all special cases of the sum $s(a,b,c)$ expressed by the general formula (3).

Corollary 1. From Theorem 1 it follows that for $a \geq 1$ and integer $d \geq 1$ it holds

$$s(a, a+d, a+2d) = \frac{H_{2d} - H_{a-1} - 2H_d + 2H_{a+d-1} - H_{a+2d-1} + 2\Psi(d+1) - \Psi(2d+1) + \gamma}{2d^2}.$$

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(a,b,c)$ for $a = 1,2,3,4$ and for $b = a+1, a+2, \dots, a+5$, $c = b+1, b+2, \dots, b+5$. We use on the one hand approximated evaluation of the sum

$$s(a,b,c,t) = \sum_{\substack{k=1 \\ k \neq a,b,c}}^{t} \frac{1}{(k-a)(k-b)(k-c)},$$

where $t = 10^7$, using the basic programming language of the computer algebra system Maple 16, and on the other hand the formula (3) for evaluation the sum $s(a,b,c)$. We compare 60 pairs of these ways obtained sums $s(a,b,c,t)$ and $s(a,b,c)$ to verify the formula (3). We use following simple procedure `rp3abcpos` and the following double repetition statement:

```
> rp3abcpos:=proc(a,b,c,t)
    local A,B,C,k,sabc,sabct; A:= a^2-a*b-a*c+b*c; B:=a*b+b*c-a*c-b^2;
    C:=a*b-a*c-b*c+c^2; sabct:=0;
    sabc:= (a+b-2*c)*(harmonic(b-a-1)-harmonic(b-a))/((a-c)*B)+((-a+2*b-c)
    *harmonic(c-a-1)-(2*a-b-c)*harmonic(c-b-1))/((a-b)*C)+(harmonic(c-a)
    +harmonic(c-b)-harmonic(c-1))/C-harmonic(a-1)/A+harmonic(b-1)/B+((a-c)
    *Psi(c+1-b)+(c-b)*Psi(c+1-a)+(a-b)*gamma)/((a-b)*(a-c)*(b-c));
    print("t=",t,"s(",a,b,c,")",evalf[12](sabc));
    for k from 1 to t do
        if k<>a then if k<>b then if k<>c then
           sabct:=sabct+1/((k-a)*(k-b)*(k-c))
        else sabct:=sabct+0; end if; end if; end if;
    end do;
    print("sum(",a,b,c,")=",evalf[12](sabct));
    print("diff=",evalf[12](abs(sabct-sabc)));
  end proc:

> for i from 1 to 4 do
    for j from i+1 to i+5 do
       for k from j+1 to i+6 do
          rp3abcpos(i,j,k,10000000);
       end do;
    end do;
  end do;
```

The approximated values of the sums $s(a,b,c)$ rounded to 10 decimals obtained by this procedure are written into the following table:

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Table 1: The approximate values of the sums $s(a,b,c)$ for $a = 1,2,3,4$, $b = a+1, a+2, \dots, a+5$ and $c = b+1, b+2, \dots, b+5$, obtained by means of the formula (3)

$\boldsymbol{a = 1}$	$c = 3$	$c = 4$	$c = 5$	$c = 6$	$c = 7$
$b = 2$	0.2500000000	−0.3611111111	−0.3263888889	−0.2691666667	−0.2238888889
$b = 3$	×	0.6944444444	0.1145833333	0.0355555556	0.0159722222
$b = 4$	×	×	0.5763888889	0.1394444444	0.0675925926
$b = 5$	×	×	×	0.4691666667	0.1256944444
$b = 6$	×	×	×	×	0.3905555556
$\boldsymbol{a = 2}$	$c = 4$	$c = 5$	$c = 6$	$c = 7$	$c = 8$
$b = 3$	0.0833333333	−0.4861111111	−0.4263888889	−0.3525000000	−0.2953174603
$b = 4$	×	0.6111111111	0.0479166667	−0.0200000000	−0.0316468254
$b = 5$	×	×	0.5263888889	0.0977777778	0.0318783069
$b = 6$	×	×	×	0.4358333333	0.0971230159
$b = 7$	×	×	×	×	0.3667460317
$\boldsymbol{a = 3}$	$c = 5$	$c = 6$	$c = 7$	$c = 8$	$c = 9$
$b = 4$	0.0416666667	−0.5194444444	−0.4541666667	−0.3763095238	−0.3161507936
$b = 5$	×	0.5861111111	0.0270833333	−0.0378571429	−0.0472718254
$b = 6$	×	×	0.5097222222	0.0834920635	0.0193783069
$b = 7$	×	×	×	0.4239285714	0.0867063492
$b = 8$	×	×	×	×	0.3578174603
$\boldsymbol{a = 4}$	$c = 6$	$c = 7$	$c = 8$	$c = 9$	$c = 10$
$b = 5$	0.0250000000	−0.5333333333	−0.4660714286	−0.3867261905	−0.3254100529
$b = 6$	×	0.5750000000	0.0175595238	−0.0461904762	−0.0546792328
$b = 7$	×	×	0.5017857143	0.0765476191	0.0132054674
$b = 8$	×	×	×	0.4179761905	0.0814153439
$b = 9$	×	×	×	×	0.3531878307

Source: own modelling in Maple 16

Computation of 60 pairs of the sums $s(a,b,c)$ and $s(a,b,c,10^7)$ took almost 2 hours. The absolute errors, i.e. the differences $|s(a,b,c) - s(a,b,c,10^7)|$, are all only about $5 \cdot 10^{-15}$.

CONCLUSIONS

We dealt with the sum of the series of reciprocals of the cubic polynomials with different positive integer roots $a < b < c$, i.e. with the series

$$\sum_{\substack{k=1 \\ k \neq a,b,c}}^{\infty} \frac{1}{(k-a)(k-b)(k-c)}.$$

We derived that the sum $s(a,b,c)$ of this series is given by formula

$$s(a,b,c) = \frac{(a+b-2c)(H_{b-a-1} - H_{b-a})}{(a-c)(ab+bc-ac-b^2)} + \frac{(-a+2b-c)H_{c-a-1} - (2a-b-c)H_{c-b-1}}{(a-b)(ab-ac-bc+c^2)}$$
$$+ \frac{H_{c-a} + H_{c-b} - H_{c-1}}{ab-ac-bc+c^2} - \frac{H_{a-1}}{a^2-ab-ac+bc} + \frac{H_{b-1}}{ab+bc-ac-b^2}$$
$$+ \frac{(a-c)\Psi(c+1-b) + (c-b)\Psi(c+1-a) + (a-b)\gamma}{(a-b)(a-c)(b-c)}.$$

We verified this main result by computing 60 sums by using the computer algebra system Maple 16. We also stated a special case of the sum $s(a,b,c)$ for $a \geq 1$ and integer $d \geq 1$:

$$s(a, a+d, a+2d) = \frac{H_{2d} - H_{a-1} - 2H_d + 2H_{a+d-1} - H_{a+2d-1} + 2\Psi(d+1) - \Psi(2d+1) + \gamma}{2d^2}.$$

The series above so belong to special types of the series, such as geometric and telescoping ones, which sums are given analytically by means of a formula.

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