{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2017(3), 1

Received: 2017-09-22
Accepted: 2017-10-19
Online published: 2017-12-04
DOI: http://dx.doi.org/10.15414/meraa.2017.03.01.17-23

Original Paper

The sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots

Radovan Potůček*

University of Defence, Faculty of Military Technology, Department of Mathematics and Physics, Brno, Czech Republic

ABSTRACT

This contribution, which is a follow-up to author's papers dealing with the sums of the series of reciprocals of quadratic polynomials with different positive integer roots, different negative integer roots, and one negative and one positive integer root, deals with the sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots. We derive the formula for the sum of these series and verify it by some examples evaluated using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers of mathematics who are teaching the topic Infinite series or as a subject matter for work with talented students.

KEYWORDS: sum of the series, harmonic number, purely imaginary conjugate roots, hyperbolic cotangent, computer algebra system Maple

JEL CLASSIFICATION: I30

INTRODUCTION AND BASIC NOTIONS

In the papers [6], [5] and [4] author dealt with the sums of the series of reciprocals of quadratic polynomials with different positive integer roots, different negative integer roots, and one negative and one positive integer root. This contribution is focused on the sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots.

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \ .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum

* Corresponding author: Radovan Potůček, University of Defence, Faculty of Military Technology Department of Mathematics and Physics, Kounicova 65, 662 10 Brno Brno, Czech Republic, e-mail: Radovan.Potucek@unob.cz

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a sum s.

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the square numbers (the *Basel problem*) is $\frac{\pi^2}{6}$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \doteq 1.644934.$$

The *nth harmonic number* H_n is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

The *hyperbolic cotangent* is defined as a ratio of hyperbolic cosine and hyperbolic sine

$$\coth x = \frac{\cosh x}{\sinh x}, \qquad x \neq 0.$$

Because (see [7]) hyperbolic cosine and hyperbolic sine can be defined in terms of the exponential function

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} + 1}{2\mathrm{e}^x}, \qquad \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} - 1}{2\mathrm{e}^x},$$

we get

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} + 1}{\mathrm{e}^{2x} - 1}, \qquad x \neq 0. \tag{1}$$

By means of the *gamma function*, which is defined via a convergent improper integral

$$\Gamma(z) = \int_0^{\infty} x^{z-1} \mathrm{e}^{-x} \mathrm{d}x,$$

is defined so called *digamma function* (see [1])

$$\Psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln(\Gamma(z)) = \frac{\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(z)}{\Gamma(z)}.$$

MATERIAL AND METHODS

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH REAL ROOTS

As regards the sum of the series of reciprocals of the quadratic polynomials with different positive integer roots a and,, i.e. with the series

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\frac{1}{(k-a)(k-b)},$$

in the paper [6] it was derived that the sum $s(a,b)$ is given by the following formula using the nth harmonic numbers H_n

$$s(a,b)=\frac{1}{b-a}\left(H_{a-1}-H_{b-1}+2H_{b-a}-2H_{b-a-1}\right). \tag{2}$$

In the paper [5] it was shown that the sum of the series of reciprocals of the quadratic polynomials with different negative integer roots a and,, i.e. with the series

$$\sum_{k=1}^{\infty}\frac{1}{(k-a)(k-b)},$$

is given by the simple formula

$$s(a,b)=\frac{1}{b-a}\left(H_{-a}-H_{-b}\right). \tag{3}$$

The sum of the series

$$\sum_{\substack{k=1\\k\neq b}}^{\infty}\frac{1}{(k-a)(k-b)}$$

of reciprocals of the quadratic polynomials with integer roots, $b>0$ was derived in the paper [4]. This sum is given by the formula

$$s(a,b)=\frac{(b-a)(H_{-a}-H_{b-1})+1}{(b-a)^2}. \tag{4}$$

THE ASSIGNMENT OF THE SOLVED PROBLEM

Now, we deal with the problem to determine the sum $s(b)$ of the series

$$\sum_{k=1}^{\infty}\frac{1}{k^2+b^2},$$

where, i.e. if the quadratic polynomial in the denominator has conjugate purely imaginary roots $k_{1,2}=\pm bi$, where is the imaginary unit. So, we can write

$$s(b) = \sum_{k=1}^{\infty} \frac{1}{k^2 + b^2}. \tag{5}$$

For example, we want to determine the sum, corresponding with the complex conjugates roots $k_{1,2} = \pm 4i$, of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 16} = \frac{1}{17} + \frac{1}{20} + \frac{1}{25} + \frac{1}{32} + \frac{1}{41} + \cdots .$$

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH PURELY IMAGINARY CONJUGATE ROOTS

Through the Weierstrass product (see [3]) for the hyperbolic sine function

$$\frac{\sinh z}{z} = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 k^2}\right),$$

where $z \in C - \{0\}$, and by the logarithmic derivation of both its sides

$$\ln \frac{\sinh z}{z} = \ln \sinh z - \ln z \;\xrightarrow{\prime}\; \frac{1}{\sinh z} \cdot \cosh z - \frac{1}{z} = \coth z - \frac{1}{z},$$

$$\ln \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 k^2}\right) = \sum_{k=1}^{\infty} \ln \frac{\pi^2 k^2 + z^2}{\pi^2 k^2} \;\xrightarrow{\prime}\; \sum_{k=1}^{\infty} \frac{\pi^2 k^2}{\pi^2 k^2 + z^2} \cdot \frac{2z}{\pi^2 k^2} = \sum_{k=1}^{\infty} \frac{2z}{\pi^2 k^2 + z^2}$$

we get the equality (see web pages [1] and [2])

$$\coth z - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{2z}{z^2 + \pi^2 k^2}.$$

For $z = \pi \cdot b, b > 0$, we have

$$\coth \pi b - \frac{1}{\pi b} = \sum_{k=1}^{\infty} \frac{2\pi b}{\pi^2 b^2 + \pi^2 k^2} = \frac{2b}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 + b^2},$$

and after multiplication by the fraction $\pi/2b$ we get the equality

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + b^2} = \frac{\pi}{2b} \coth \pi b - \frac{1}{2b^2}. \tag{6}$$

By the relation (1) we get

Theorem 1 The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + b^2},$$

where $b > 0$, has the sum

$$\sum_{k=1}^{\infty}\frac{1}{k^2+b^2}=\frac{\pi}{2b}\cdot\frac{e^{2\pi b}+1}{e^{2\pi b}-1}-\frac{1}{2b^2}. \tag{7}$$

Example 1 Evaluate the sum $s(4)$ of the series

$$\sum_{k=1}^{\infty}\frac{1}{k^2+4^2}=\frac{1}{17}+\frac{1}{20}+\frac{1}{25}+\frac{1}{32}+\frac{1}{41}+\cdots$$

i) by formula (7),
ii) in the CAS Maple 16 by means of the partial sum using the first one million terms. Compare the obtained results.

Solution

i) The series has by formula (7) from Theorem 1, where, the sum

$$s(4)=\frac{\pi}{2\cdot 4}\cdot\frac{e^{2\pi\cdot 4}+1}{e^{2\pi\cdot 4}-1}-\frac{1}{2\cdot 4^2}=\frac{\pi}{8}\cdot\frac{e^{8\pi}+1}{e^{8\pi}-1}-\frac{1}{32}\doteq 0.3614490817.$$

ii) By using the CAS Maple 16 and evaluation the approximate value $s_{10^6}(4)$ of the sum $s(4)$ we get the following sequence of commands and results:

```
> sum(1/(k^2+16),k=1..1000000);
```

$$-\frac{1}{8}I\Psi(1000001-4I)+\frac{1}{8}I\Psi(1000001+4I)+\frac{1}{8}I\Psi(1-4I)-\frac{1}{8}I\Psi(1+4I)$$

```
> evalf(%,10);
```

$$0.3614480818+0.I$$

So we can state that both these obtained results, using ten first decimals, are almost the same and they differ from each other only about 10^{-6}. Let us note that in the Maple notation the symbol means the imaginary unit and the function Ψ denotes the digamma function.

RESULTS AND DISCUSSION - NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(b)$ of the series

$$\sum_{k=1}^{\infty}\frac{1}{k^2+b^2}$$

for. We use on the one hand an approximative direct evaluation of the sum

$$s(b,t)=\sum_{k=1}^{t}\frac{1}{k^2+b^2},$$

where $t=5\cdot 10^5$, using the basic programming language of the CAS Maple 16, and on the other hand formula (7) for evaluation the sum $s(b)$.

We compare 10 pairs of these two ways obtained sums $s(b,5\cdot10^5)$ and $s(b)$ to verify formula (7). Let us note that the evaluation of the sums $s(b,10^6)$ in this way would be too time consuming and would take dozens of hours. For computation the sums $s(b,5\cdot10^5)$ and $s(b)$ we use following simple procedure `sumb` and one **for** statement:

```
> sumb:=proc(b,t)
      local f,k,r,s;
      s:=0;
      r:=0;
      for k from 1 to t do
          r:=1/(k^2+b^2);
          s:=s+r;
      end do;
      print("s(",b,")=",evalf[10](s));
      f:=Pi/(2*b)*(exp(2*Pi*b)+1)/(exp(2*Pi*b)-1)-1/(2*b^2);
      print("f(",b,")=",evalf[10](f));
      print("abserr=",evalf[10](abs(f-s)));
      print("relerr=",evalf[10](abs((f-s)/f)));
  end proc:
> for b from 1 to 10 do
      sumb(500000,b);
  end do;
```

This computation of the sums $s(b,5\cdot10^5)$ and $s(b)$ took almost 17 hours while the following direct computation of the sums $s(b,10^6)$ without using the programming language took only under 1 second:

```
> for b from 1 to 10 do
    print("s(",b,")=",evalf[10](sum(1/(k^2+b^2),k=1..1000000)));
  end do;
```

The approximative values of the sums, $s(b,5\cdot10^5)$, and obtained by formula (7), by the procedure `sumb` and by direct computation and rounded to 9 decimals, are written into the following table 1.

Tab.1: Some approximative values of the sums, $s(b,5\cdot10^5)$

b	1	2	3	4	5
$s(b)$	1.076674048	0.660403642	0.468043227	0.361449082	0.294159265
$s(b,5\cdot10^5)$	1.076672047	0.660401641	0.468041227	0.361447082	0.294157265
$s(b,10^6)$	1.076673047	0.660402642	0.468042227	0.361448082	0.294158265
b	6	7	8	9	10
$s(b)$	0.247910499	0.214195394	0.188537041	0.168360086	0.152079633
$s(b,5\cdot10^5)$	0.247908499	0.214193394	0.188535041	0.168358086	0.152077633
$s(b,10^6)$	0.247909499	0.214194394	0.188536041	0.168359086	0.152078633

The relative errors, i.e. the ratios $\left|\frac{s(b)-s(b,5\cdot 10^5)}{s(b)}\right|$, range between $2\cdot 10^{-6}$ for $b=1$ and 10^{-5} for.

CONCLUSIONS

We dealt with the sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots $\pm bi$, i.e. with the series $\sum_{k=1}^{\infty}\frac{1}{k^2+b^2}$.

By means of the Weierstrass product and logarithmic derivation we derived that the sum $s(b)$ of this series is given by the formula

$$s(b) = \frac{\pi}{2b}\coth \pi b - \frac{1}{2b^2},$$

$$s(b) = \frac{\pi}{2b}\cdot\frac{\mathrm{e}^{2\pi b}+1}{\mathrm{e}^{2\pi b}-1} - \frac{1}{2b^2}.$$

We verified this main result by computing 10 sums by using the CAS Maple 16.

The series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a formula which can be expressed in closed form.

ACKNOWLEDGEMENT

The work presented in this paper has been supported by the project „Rozvoj oblastí základního a aplikovaného výzkumu dlouhodobě rozvíjených na katedrách teoretického a aplikovaného základu FVT (K215, K217)“.

REFERENCES

[1] *Mathematics Stack Exchange – A question and answer website for people studying math*. (2017). Retrieved 2017-08-31 from https://math.stackexchange.com/questions/736860/find-the-infinite-sum-of-the-series-sum-n-1-infty-frac1n2-1

[2] *Mathematics Stack Exchange – A question and answer website for people studying math*. (2017). Retrieved 2017-08-31 from https://math.stackexchange.com/questions/742855/help-with-calculating-infinite-sum-sum-n-0-infty-frac11n2?noredirect=1&lq=1

[3] Melnikov, Y. A. (2011). *Green's Functions and Infinite Product. Bridging the Divide. Chapter 2: Infinite Products and Elementary Functions*, pp. 17-42. doi:http://dx.doi.org/10.1007/978-0-8176-8280-4

[4] Potůček, R. (2017). The sum of the series of reciprocals of the quadratic polynomial with one negative and one positive integer root. In *Proceedings of the 16th Conference on Applied Mathematics APLIMAT 2017*, Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, 2017, pp. 1252-1253.

[5] Potůček, R. (2016). The sum of the series of reciprocals of the quadratic polynomial with different negative integer roots. *Ratio Mathematica - Journal of Foundations and Applications of Mathematics*, vol. 30(1), 59-66.

[6] Potůček, R. (2016). The sum of the series of reciprocals of the quadratic polynomial with different positive integer roots. In *Mathematics, Information Technologies and Applied Sciences 2016 (MITAV 2016)*: Conference proceedings of extended versions of selected papers, Brno: Univerzita obrany, 2016, pp. 71-83.