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Original Paper

The sum of the series of reciprocals of the quadratic polynomials with double positive integer root

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ABSTRACT

This contribution, which is a follow-up to author's papers [3] and [4], deals with the series of reciprocals of the quadratic polynomials with double positive integer root. The formula for the sum of this kind of series expressed by means of harmonic numbers are derived and verified by several examples evaluated using the basic programming language of the computer algebra system Maple 16. There is stated another formula using generalized harmonic numbers, too. This contribution can be an inspiration for teachers of mathematics who are teaching the topic Infinite series or as a subject matter for work with talented students.

KEYWORDS: telescoping series, harmonic numbers, CAS Maple, Riemann zeta function

JEL CLASSIFICATION: I30

INTRODUCTION AND BASIC NOTIONS

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \ .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \ .$$

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The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$.

The n-th *harmonic number* is the sum of the reciprocals of the first n natural numbers: $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n = \sum_{k=1}^{n} 1/k$. The *generalized harmonic number* of order n in power r is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r}, \tag{1}$$

where $H_{n,1} = H_n$ are harmonic numbers. Every generalized harmonic number of order n in power m can be written as a function of generalized harmonic number of order n in power $m-1$ using formula (see [6])

$$H_{n,m} = \sum_{k=1}^{n-1} \frac{H_{k,m-1}}{k(k+1)} + \frac{H_{n,m-1}}{n}, \tag{2}$$

whence

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n}. \tag{3}$$

From formula (1), where $r = 1,2$ and $n = 1,2,\dots,10$, we get the following table 1.

Tab.1: Some harmonic and generalized harmonic numbers

n	1	2	3	4	5	6	7	8	9	10
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$
$H_{n,2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	$\frac{1077749}{705600}$	$\frac{771817}{352800}$	$\frac{1968329}{1270080}$

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH DOUBLE POSITIVE INTEGER ROOT

We deal with the problem to determine the sum $s(a,a)$ of the series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{(k-a)^2}$$

for positive integers a, i.e. to determine the sum $s(1,1)$ of the series

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots, \tag{4}$$

the sum $s(2,2)$ of the series

$$\sum_{\substack{k=1\\k\neq 2}}^{\infty}\frac{1}{(k-2)^2}=\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=s(1,1)+1\,,$$

the sum $s(3,3)$ of the series

$$\sum_{\substack{k=1\\k\neq 3}}^{\infty}\frac{1}{(k-3)^2}=\frac{1}{2^2}+\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=s(1,1)+\left(1+\frac{1}{2^2}\right)=s(1,1)+\frac{5}{4}\,,$$

etc. Clearly, we get the formula

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2}=s(1,1)+s_{a-1}(1,1)\,, \tag{5}$$

where $s_{a-1}(1,1)$ is the $(a-1)$th partial sum of the series (4), and also the formula

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2}=2s_{a-1}(1,1)+\sum_{k=a}^{\infty}\frac{1}{k^2}\,. \tag{6}$$

A problem to determine the sum $s(1,1)=1/1^2+1/2^2+1/3^2+1/4^2+\cdots$ is so called Basel problem. This problem was posed by *Pietro Mengoli* (1625-1686) in 1644. In 1689 Jacob Bernoulli (1654-1705) proved that the series $\sum_{k=1}^{\infty}1/k^2$ converges and its sum is less than 2. In 1737 *Leonhard Euler* (1707-1783) showed his famous result $\sum_{k=1}^{\infty}1/k^2=\pi^2/6$. This sum presents the value $\zeta(2)$ of the *Riemann zeta function*

$$\zeta(s)=\sum_{k=1}^{\infty}k^{-s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\ .$$

The values of the n-th partial sum $s_n(1,1)=1/1^2+1/2^2+1/3^2+\cdots+1/n^2$ correspond to the values $H_{n,2}$, so their first ten values are presented in the third row of the table 1. Some another values of the n-th partial sums $s_n(1,1)$, computed by CAS Maple 16, are $s_{100}\doteq 1.634984$, $s_{1000}\doteq 1.643935$, $s_{10000}\doteq 1.644834$, $s_{100000}\doteq 1.644924$, $s_{1000000}\doteq 1.644933$, whereas the series $s(1,1)$ converges to the number $1.644934066\ldots$.

The partial sums $s_n(1,1)$, i.e. generalized harmonic numbers $H_{n,2}$ are also determined by the formula (see [5])

$$s_n(1,1)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(i-1)!\,(j-1)!}{(i+j)!}+\frac{3}{2}\sum_{l=1}^{n}\frac{1}{l^2\binom{2l}{l}}\,. \tag{7}$$

This surprising identity was derived by the contemporary brilliant amateur French mathematician Benoit Cloitre (see [1]).

According to formula (5) is

$$s(a,a)=\zeta(2)+H_{a-1,2}\,. \tag{8}$$

Using formulas (5) and (7) we get

Theorem 1 The series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{(k-a)^2},$$

where $a > 0$ is integer, has the sum

$$s(a,a) = \frac{\pi^2}{6} + \frac{1}{2}\left[\sum_{i=1}^{a-1}\sum_{j=1}^{a-1}\frac{(i-1)!\,(j-1)!}{(i+j)!} + 3\sum_{l=1}^{a-1}\frac{1}{l^2\binom{2l}{l}}\right]. \tag{9}$$

Remark 1 In [2] it is stated the equality

$$\int_0^{\infty}\frac{te^{-Nt}}{e^t-1}\,\mathrm{d}t = \sum_{k=N+1}^{\infty}\frac{1}{k^2}, \tag{10}$$

which can be proved using a geometric sum-type expansion of the denominator and evaluation of the subsequent integrals by means of the integration by parts and L'Hôpital's Rule.

Using formula (6) we get

Theorem 2 The series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{(k-a)^2},$$

where $a > 0$ is integer, has the sum

$$s(a,a) = 2H_{a-1,2} + \int_0^{\infty}\frac{te^{-(a-1)t}}{e^t-1}\,\mathrm{d}t\,. \tag{11}$$

Example 1 Evaluate the sum of the series

$$\sum_{\substack{k=1 \\ k\neq 5}}^{\infty} \frac{1}{(k-5)^2} = \frac{1}{4^2}+\frac{1}{3^2}+\frac{1}{2^2}+\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

by formula **i)** (9), **ii)** (11), and **iii)** (8) and compare the obtained results.

Solution

i) The series has by formula (9) from Theorem 1, where $a = 5$, the sum

$$s(5,5) = \frac{\pi^2}{6} + \frac{1}{2}\left[\sum_{i=1}^{4}(i-1)!\sum_{j=1}^{4}\frac{(j-1)!}{(i+j)!} + 3\sum_{l=1}^{4}\frac{1}{l^2\binom{2l}{l}}\right] = \frac{\pi^2}{6} + \frac{S}{2}.$$

At first, we successive evaluate the sum S. We get

$$S = 0!\left(\frac{0!}{2!}+\frac{1!}{3!}+\frac{2!}{4!}+\frac{3!}{5!}\right)+1!\left(\frac{0!}{3!}+\frac{1!}{4!}+\frac{2!}{5!}+\frac{3!}{6!}\right)+2!\left(\frac{0!}{4!}+\frac{1!}{5!}+\frac{2!}{6!}+\frac{3!}{7!}\right)$$
$$+3!\left(\frac{0!}{5!}+\frac{1!}{6!}+\frac{2!}{7!}+\frac{3!}{8!}\right)+3\left(\frac{1}{1\cdot\binom{2}{1}}+\frac{1}{4\cdot\binom{4}{2}}+\frac{1}{9\cdot\binom{6}{3}}+\frac{1}{16\cdot\binom{8}{4}}\right)=\frac{205}{72}.$$

Now, we have

$$s(5,5)=\frac{\pi^2}{6}+\frac{1}{2}\cdot\frac{205}{72}\doteq 3.0685451780\,.$$

ii) By formula (11) from Theorem 2, where $a=5$, by means of integration using Maple 16, we also get the required sum:

$$s(5,5)=2H_{4,2}+\int_0^{\infty}\frac{te^{-4t}}{e^t-1}\mathrm{d}t\doteq 2\cdot\frac{205}{144}-0.2211322959\doteq 3.0685451809\,.$$

iii) The third and much more easily way, how to determine the sum $s(5,5)$, is to use formula (8) and the value of $H_{a-1,2}=H_{4,2}=205/144$ from the table 1. So we immediately obtain the required result:

$$s(5,5)=\zeta(2)+H_{4,2}=\frac{\pi^2}{6}+\frac{205}{144}\doteq 3.0685451780\,.$$

Formulas (9) and (8) give the identical result 3.06854517807 and the result obtained by formula (11) differs from them only about $3\cdot 10^{-9}$.

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(a,a)$ of the series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2}$$

for $a=1,2,\ldots,10,50,99,100,500,999,1000$. We use on the one hand an approximative direct evaluation of the sum

$$s(a,a,t)=\sum_{\substack{k=1\\k\neq a}}^{t}\frac{1}{(k-a)^2}\,,$$

where $t=10^6$, using the basic programming language of the CAS Maple 16, and on the other hand the formula (9) for evaluation the sum $s(a,a)$. We compare 16 pairs of these two ways obtained sums $s(a,a,10^6)$ and $s(a,a)$ to verify formula (9). We use following simple procedure rp2raapos and one for statement:

```
rp2raapos:=proc(a,t)
    local i,j,A,s1,s2,saa,sumaa;
    A:=a-1; s1:=0; s2:=0; saa:=0; sumaa:=0;
    for i from 1 to A do
        for j from 1 to A do
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            s1:=s1+((i-1)!*(j-1)!)/((i+j)!);
        end do;
    end do;
    for i from 1 to A do
        s2:=s2+1/(i*i*binomial(2*i,i));
    end do;
    saa:=Pi*Pi/6+s1/2+3*s2/2;
    print("a=",a,": saa=",evalf[20](saa));
    for i from 1 to t do
        if i <> a then
        sumaa:=sumaa+1/((i-a)*(i-a));
        end if;
    end do;
    print("sumaa(",t,")=",evalf[20](sumaa));
    print("diff=",evalf[20](abs(sumaa-saa)));
end proc:
for a in [1,2,3,4,5,6,7,8,9,10,50,99,100,500,999,1000] do
    rp2raapos(a,1000000);
end do;
```

The approximative values of the sums $s(a,a)$ obtained by the procedure rp2raapos and rounded to 6 decimals, are written into the following table 2.

Tab. 2 Some approximative values of the sums $s = s(a,a)$

a	1	2	3	4	5	6	7	8
s	1.644934	2.644934	2.894934	3.006045	3.068545	3.108545	3.136323	3.156731
a	9	10	50	99	100	500	999	1000
s	3.172356	3.184701	3.269667	3.279716	3.279818	3.287866	3.288867	3.288868

Computation of 16 couples of the sums $s(a,a,10^6)$ and $s(a,a)$ took over 16 hours. The relative errors, i.e. the ratios $|[s(a,a) - s(a,a,10^6)]/s(a,a)|$, range between 10^{-7} for $a = 1$ and 10^{-9} for $a = 1000$.

CONCLUSIONS

We dealt with the sum of the series of reciprocals of the quadratic polynomials with double positive integer root a, i.e. with the series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{(k-a)^2} \, .$$

We derived that the sum $s(a,a)$ of this series is given by the formula

$$s(a,a) = \frac{\pi^2}{6} + \frac{1}{2}\left[\sum_{i=1}^{a-1}\sum_{j=1}^{a-1} \frac{(i-1)!\,(j-1)!}{(i+j)!} + 3\sum_{l=1}^{a-1} \frac{1}{l^2\binom{2l}{l}}\right].$$

We verified this main result by computing 16 sums by using the CAS Maple 16.

Two another ways how to calculate the sum $s(a,a)$ is using the value of generalized harmonic number $H_{a-1,2}$ of order $a-1$ in power 2 and the improper integral

$$s(a,a) = 2H_{a-1,2} + \int_0^\infty \frac{te^{-(a-1)t}}{\mathrm{e}^t - 1}\mathrm{d}t$$

or the short formula with the value of the generalized harmonic number $H_{a-1,2}$

$$s(a,a) = \frac{\pi}{2} + H_{a-1,2} \ .$$

The series of reciprocals of the quadratic polynomials with double positive integer root so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a formula which can be expressed in closed form.

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