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Original paper

Derivations of the Bézier Curve

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ABSTRACT

On selected polygon of control points we create Bézier curve. This we differentiate using a direct derivation of the polygon and receive hodograph of a new Bézier curve. The process is repeated for the second derivation, and these results are compared with numerical derivation of the parametric specified curve.

KEYWORDS: tangent curve, hodograph, numerical derivation

JEL CLASSIFICATION: M55, N55

INTRODUCTION

Bézier curve needs for its definition a set of control points that determine it completely [3]. Its application, as well as the use of B-spline curve we addressed our previous works [2]. In finding tangents and normals of Bézier curve is necessary to calculate its derivation [1], [3]. Similarly, the first and the second derivation is used in determining of the length and curvature of any curve. The article briefly describes method for obtaining the derivations of a Bézier curve and on a simple example this method is compared with the numerical derivation for the parametric specified curve.

MATERIAL AND METHODS

Derivations of a Bézier curve

Bézier curve is defined by the polygon of (n+1) control points P_0, P_1, ... , P_n and follows the equation

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$$P(t)=\sum_{i=0}^{n} B_i^n(t).P_i \tag{1}$$

where Bernstein polynomial is defined

$$B_i^n(t)=\binom{n}{i}.t^i.(1-t)^{n-i} \tag{2}$$

The control points P_i are independent on parameter t and therefore is sufficient to differentiate $B_i^n(t)$

$$\frac{d}{dt}B_i^n(t)=B_i^{n\prime}(t)=n.(B_{i-1}^{n-1}(t)-B_i^{n-1}(t)) \tag{3}$$

Then the derivation throughout the Bézier curve has the form

$$\frac{d}{dt}P(t)=P'(t)=\sum_{i=0}^{n-1} B_i^{n-1}(t).\left[n.(P_{i+1}-P_i)\right] \tag{4}$$

If we introduce a set of control points $Q_0 = n(P_1 - P_0)$, $Q_1 = n(P_2 - P_1)$, $Q_2 = n(P_3 - P_2)$, ..., $Q_n = n(P_n - P_{n-1})$ writing of derivation (4) simplify on

$$P'(t)=\sum_{i=0}^{n-1} B_i^{n-1}(t).Q_i \tag{5}$$

When (5) compared to (1) we see that by differentiation of the Bézier curve arises again Bézier curve of degree (n-1), with control points $n(P_1 - P_0)$, $n(P_2 - P_1)$, $n(P_3 - P_2)$, ... , $n(P_n - P_{n-1})$. This new differentiated curve is usually called the hodograph of the original Bézier curve. The form $n(P_{i+1} - P_i)$ represents n-times direction vector from the point P_i to the point P_{i+1}.

For t = 0, $P'(0) = n(P_1 - P_0)$, which means that the first tangent of the curve is in the direction of $(P_1 - P_0)$ n-times multiplied. Similarly for t = 1, $P'(1) = n(P_n - P_{n-1})$, the last tangent of the curve is in the direction of $(P_n - P_{n-1})$ n-times enlarged.

In calculating the second derivation we start from equation (5) and again analogously we differentiate as for the first derivation

$$\begin{aligned} P''(t)&=\sum_{i=0}^{n-2} B_i^{n-2}(t).\left[(n-1).(Q_{i+1}-Q_i)\right] \\ &=\sum_{i=0}^{n-2} B_i^{n-2}(t).\left[n.(n-1).(P_{i+2}-2P_{i+1}+P_i)\right] \end{aligned} \tag{6}$$

The higher derivations can be obtained likewise by applying the same process.

RESULTS AND DISCUSSION

We chose ten points in the plane, n = 9, where x and y-coordinates are given in the Table 1. These control points have been plotted and we constructed the Bézier curve according to [2]. The result is shown in the Figure 1.

By derivation of the Bézier curve (4) we obtained its hodograph drawn in the Figure 2. The control points of the new curve are calculated in the middle two columns of the Table 1. Degree of the new Bézier curve was reduced.

The process was repeated and it was found the second derivation of the original curve according to (6). The graph is shown in the Figure 3 and the coordinates of the control points are given in the Table 1 in the last two columns. Degree of the Bézier curve of second derivation again decreased.

The values of both obtained derivations are presented in the Table 2 for selected parameter step t = 0.01. Each curve is parameterized through t∈<0, 1>. From all 101 values we point out every tenth value.

Tab. 1 Coordinates of control points of a Bézier curve and its first and second derivations

Bézier curve, n = 9			1st derivation, n = 8		2nd derivation, n = 7	
i	$P_i[x]$	$P_i[y]$	$P_i'[x]$	$P_i'[y]$	$P_i''[x]$	$P_i''[y]$
0	14	0	-72	36	-144	576
1	6	4	-90	108	1296	-648
2	-4	16	72	27	144	576
3	4	19	90	99	-576	-792
4	14	30	18	0	576	-792
5	16	30	90	-99	-2232	576
6	26	19	-189	-27	2880	-648
7	5	16	171	-108	-1944	576
8	24	4	-72	-36	-	-
9	16	0	-	-	-	-

Tab. 2 Points of Bézier curve and curves forming the first and second derivations

	Bézier curve		1st derivation		2nd derivation	
	P[x]	P[y]	P´[x]	P´[y]	P´´[x]	P´´[y]
0	14	0	-72	36	-144	576
10	7.357225	5.402068	-51.62585	64.071936	418.985914	85.74912
20	4.450247	11.921203	-6.478664	63.790848	426.933965	-69.76512
30	5.652148	17.775412	27.243613	51.558912	233.447731	-174.19392
40	9.176722	21.894451	39.682437	29.263104	20.73047	-266.84928
50	12.960938	23.390625	33.46875	0	-130.5	-306
60	15.531450	21.894451	17.232261	-29.263104	-172.40279	-266.84928
70	16.497654	17.775412	3.741085	-51.558912	-74.676211	-174.19392
80	16.783494	11.921203	4.746424	-63.790848	85.544755	-69.76512
90	17.649271	5.402068	9.410566	-64.071936	-123.408634	85.74912
100	16	0	-72	-36	-1944	576

For the parameter t = 0.01, in every tenth value, obtained by direct calculation

Tab. 3 Points of Bézier curve and curves forming the first and second derivations

Bézier curve			1st derivation		2nd derivation	
	P[x]	P[y]	P´[x]	P´[y]	P´´[x]	P´´[y]
0	14	0	-72.5565	38.7414	-49.39	496.06
10	7.357225	5.402068	-49.4982	64.4611	436.96	62.87
20	4.450247	11.921203	-4.3674	63.4239	412.36	-80.57
30	5.652148	17.775412	28.3737	50.6708	211.19	-184.46
40	9.176722	21.894451	39.7546	27.9168	2.01	-273.87
50	12.960938	23.390625	32.7993	-1.5297	-140.23	-305.51
60	15.531450	21.894451	16.3753	-30.5848	-168.88	-259.18
70	16.497654	17.775412	3.3947	-52.4127	-58.19	-163.85
80	16.783494	11.921203	5.1888	-64.1212	93.19	-58.60
90	17.649271	5.402068	8.6690	-63.6011	-203.46	112.09
Last	16	0	-62.8257	-38.7414	-1629.74	496.06

For the parameter t = 0.01, in every tenth value, obtained by numerical calculation

Since the Bézier curves for a set of control points are defined parametrically, can be differentiated also numerically. The results of these numerical derivations are given in the Table 3. The last line in this table identified Last means for the original curve i = 100, for the first derivation i = 99 and for the second derivation i = 98. The number of points is reduced, which directly follows from numerical differentiation algorithm.

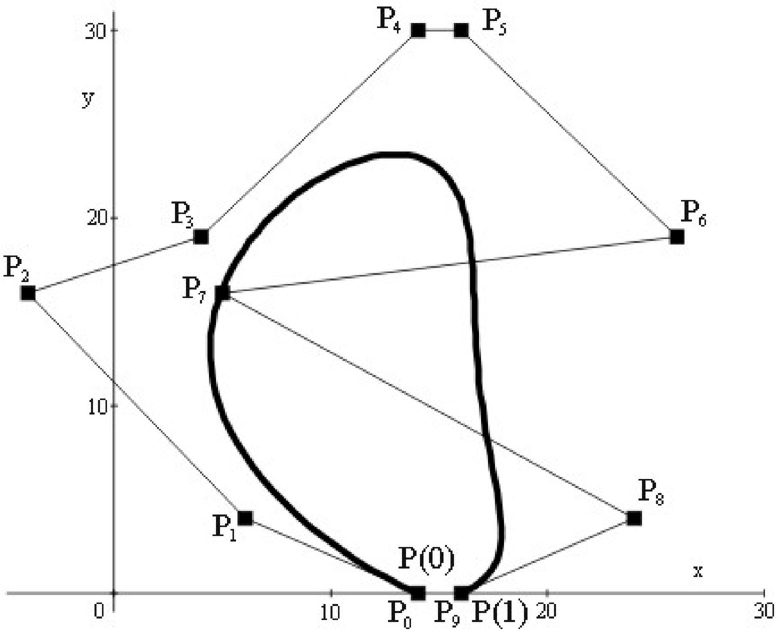

Fig. 1 Bézier curve and its control points drawn for the parameter t = 0.01

Obtained curve of the first derivation is shown in the Figure 4 and the second derivation curve obtained is in the Figure 5.

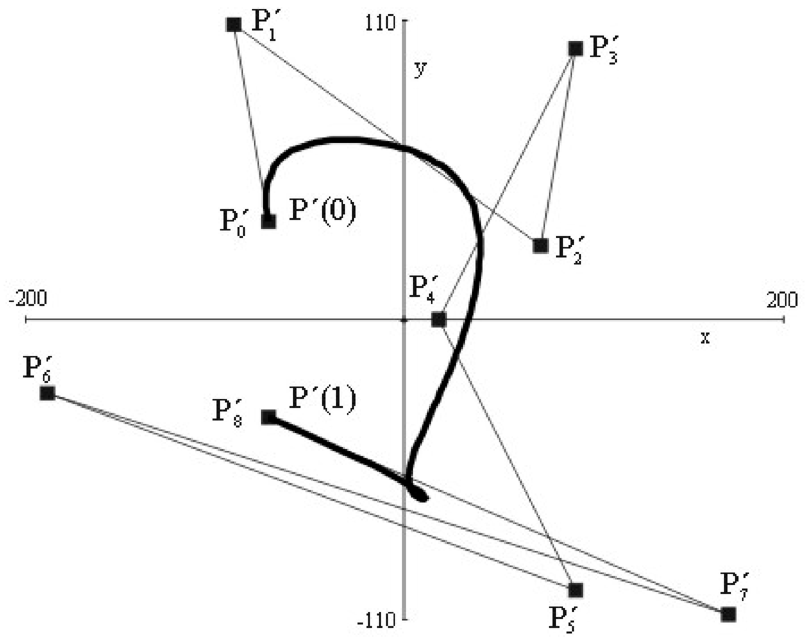

Fig. 2 The first derivation of the Bézier curve with its control points
For the parameter t = 0.01, obtained by direct calculation

Comparing the pairs of the Figures 2 and 4 we see that result of the first derivation found by direct calculation (5) corresponds to the result of numerical derivation. Similar finding belongs to the comparison of the second derivations in the Figures 3 and 5.

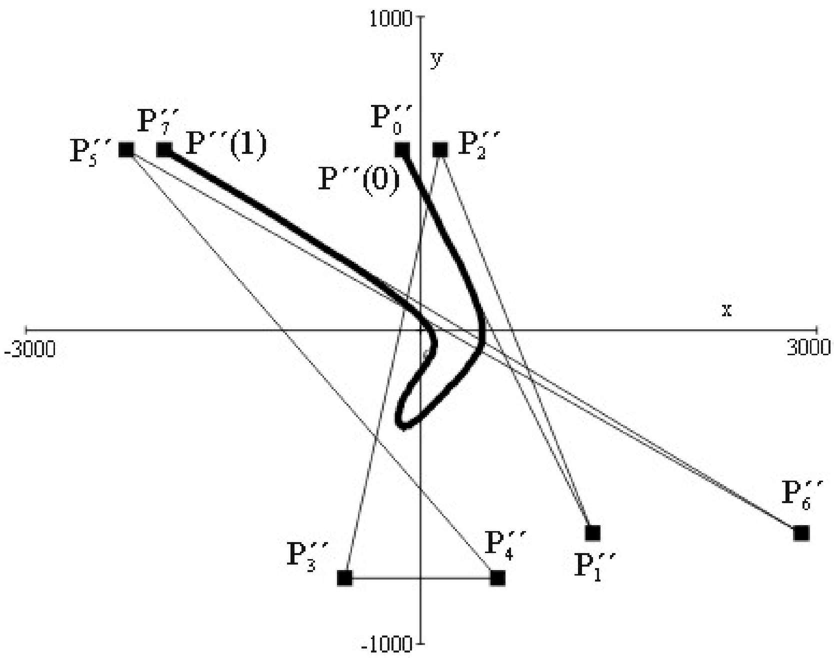

Fig. 3 The second derivation of the Bézier curve with its control points
For the parameter t = 0.01, obtained by direct calculation

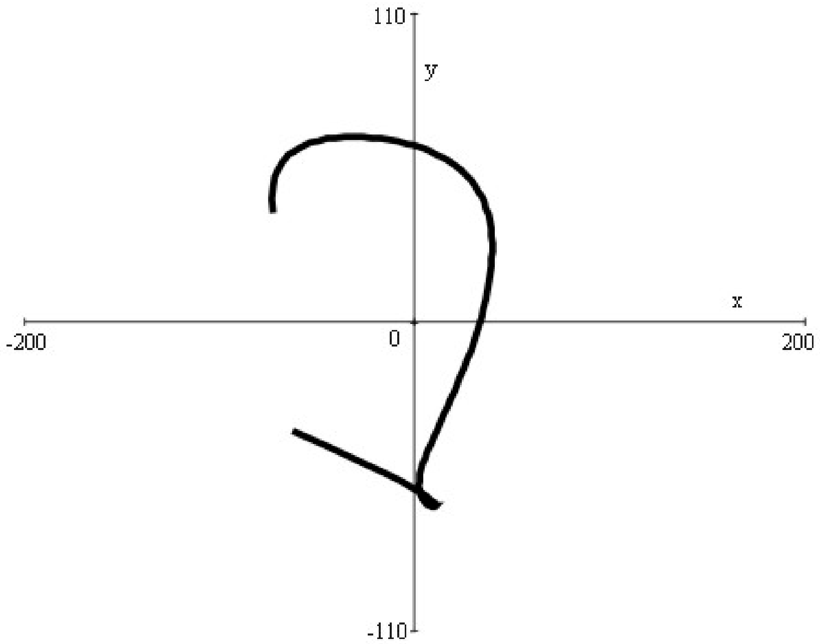

Fig. 4 The first derivation of the Bézier curve with its control points
For the parameter t = 0.01, obtained by numerical calculation

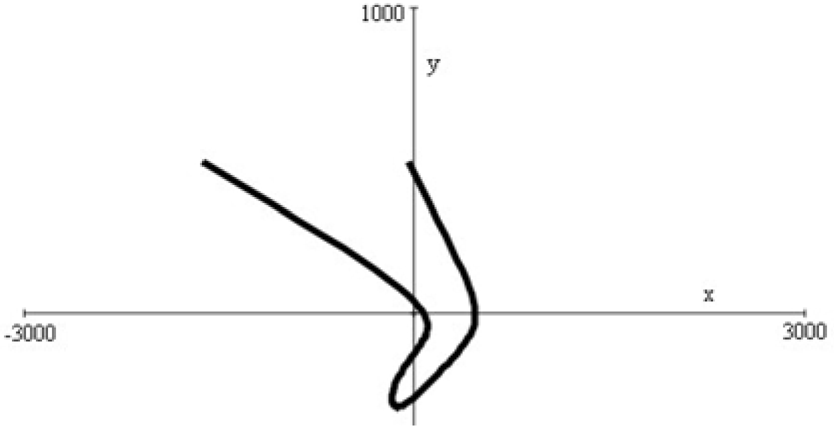

Fig. 5 The second derivation of the Bézier curve with its control points.
For the parameter t = 0.01, obtained by numerical calculation

When comparing the values of the first and second derivation in the Tables 2 and 3 there is not already excessive identity and mainly the second derivations are quite different. However, it should be noted scales of the Figures 1 - 5. They result that with increasing derivation also scale is increasing, which is a direct consequence of the relationships (4) and (5). Therefore also relatively large differences between direct and numerical derivation significantly do not affect the outcome and shape of the curve remains the same by both methods.

CONCLUSIONS

We created Bézier curve and its first two derivations, which again formed a new Bézier curves. This direct calculation of the derivations was compared with the numerical derivations of the original curve. The results in the Figures 2 - 5 are coincided for both derivatives. By comparing the respective values in the Tables 2 and 3 resulting differences were caused by inaccuracy of the numerical derivation. The inaccuracy increases with degree of derivation. Nevertheless numerical derivation very well approximated a direct derivation of the Bézier curve.

REFERENCES

[1] *A primer on Bézier curves: A free, online book for when you really need to know how to do Bézier things*. Retrieved 2016-15-02 from http://pomax.github.io/bezierinfo/

[2] Páleš D. & Rédl J. (2015). Bézier curve and its application. *Mathematics in Education, Research and Applications*, 1(2), 49-55. doi:http://dx.doi.org/10.15414/meraa.2015.01.02.49-55

[3] Shene, C.-K. (2014). *Introduction to Computing with Geometric Notes*. Retrieved 2016-15-02 from http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/