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Original paper

Are proofs necessary?

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ABSTRACT
Proofs of mathematical theorems are vital part of mathematics as a science, and mathematics as a university subject as well. Students are not used to such an approach, which is a source of many unexpected problems for them. The submitted contribution aims to justify usefulness of mathematical proofs presenting them on proofs of simple statements, and thus explaining their necessity during the whole university studies.

KEYWORDS: proof, (ir)rational number, real numbers

JEL CLASSIFICATION: E 10

INTRODUCTION

Mathematics is a science with strictly deductive structure. Its basis lies in axioms, i. e. propositions which, taking into account general mathematical experience, are considered true, or in other words, are not questioned in terms of their veracity (e. g. commutative law of real numbers addition). Mathematical objects are firstly defined in form of mathematical definitions. Secondly, they are studied in terms of their properties. When mathematicians discover a property of a mathematical object, they formulate it as a mathematical theorem. However, foundation stones of any theory within mathematics cannot be stable without proving mathematical theorems, establishing the truth of the statements, i. e. verification.

Learners first encounter this approach towards mathematics education at universities and colleges, usually in calculus lectures. Until that moment they perceive mathematical proofs only as one of those high school topics which they had forgotten long before. Yet, in their upcoming studies students are going to regularly encounter direct proofs, proofs by

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contrapositive, proofs by contradiction, and proofs by mathematical induction. Therefore, for students proofs are going to become inevitable and at the same time the most disliked part of mathematics curriculum, and not only in calculus lessons.

MATERIAL AND METHODS

The submitted article seeks to show the importance of proofs in mathematical theory. Below, authors are dealing not only with simple propositions, which students have considered obvious for years, so obvious and trivial that students had no need to think about them more deeply. The authors of the article try to show that mathematical proofs are crucial in justifying veracity and usefulness of the theory in question, for in pure mathematics, unlike in politics, nothing can be declared without being verified.

RESULTS AND DISCUSSION

Rational, irrational and real numbers

One of the key concepts in mathematics is a set. A set is a collection of objects (elements), which can be directly named, or defined by their common characteristic property. A special type of a set is an interval, and an arbitrary interval will be discussed further on in this article. One of those simple statements which will help us explain the importance of proofs in mathematics lessons is the following theorem.

Theorem: Let $a, b \in \mathbf{R}$*. Then there are infinitely many rational and infinitely many irrational numbers between a and b.*

As we indicated above, we have really chosen a fact which is pretty known to every high school graduate. Before we get to the proof itself, let us ask a few questions which will lead us to the theorem.
Are there actually any rational and irrational numbers? It should be no problem to find at least one rational number. The set of all rational numbers consists of all numbers in form $\frac{p}{q}$, where $p \in \mathbf{Z}$, $q \in \mathbf{N}$ and at the same time the greatest common divisor of the two numbers *p, q* is 1. So, for example, such a number is $\frac{22}{7} \in \mathbf{Q}$.

It is also no difficulty to find at least one irrational number; we state that $\sqrt{2} \in \mathbf{I}$. Now, here comes the moment when one must not believe everything other people say. It is necessary to *prove* that $\sqrt{2} \notin \mathbf{Q}$ (which will be not done right now, since later on we use a more general approach in proving the theorem in question).

Another logic question is if there are infinitely many rational and irrational numbers. Well, we can be sure that there are infinitely many natural numbers. Furthermore, set of all natural numbers is a subset of all rational numbers ($\mathbf{N} \subset \mathbf{Q}$), which means that there must be infinitely many rational numbers, too. In order to be able to examine the amount of irrational numbers, authors of the submitted article recommend that the readers have a look at the brilliant Euclid's proof of the fact that the number of primes is infinite. Having successfully studied that elegant justification, the readers can continue with the following lemma.

Lemma. Let p be a prime number. Then $\sqrt{p}$ *is an irrational number.*
Proof. Authors suggest that a proof by contradiction might be used. Assume that $\sqrt{p} \in \mathbf{Q}$. Then $\sqrt{p} = \frac{m}{n}$, where $m, n \in \mathbf{N}$ and at the same time the greatest common divisor of numbers *m* and *n* is 1 (i. e. numbers *m* and *n* indivisible). Squaring and multiplication by the denominator gives $pn^2 = m^2$, which means that m^2 must be a multiple of *p*. However, *p* is a prime, thus *m* must be a multiple of *p* as well, so for example $m = kp$, where $k \in \mathbf{Z}$. Then $m^2 = k^2 p^2$, and also $pn^2 = k^2 p^2$, hence $n^2 = k^2 p$. The last notation means that n^2 is a multiple of *p*, and that *n* is a multiple of *p*, too. Consecutively, *p* (*p*>1) is a common divisor of *m* and *n*, which is a contradiction to the initial assumption that *m* and *n* are indivisible. Hence, the assumption that $\sqrt{p} \in \mathbf{Q}$ must be false. In other words, $\sqrt{p}$ must be irrational, which is what we set out to prove.

Thus, basing on proved statements rather than impressions, we can see that there really are infinitely many rational and irrational numbers. This, however, does not necessarily mean that there are infinitely many of such numbers between *a* and *b*, which we state in our theorem. Assuming that $a < b$, we claim that there are infinitely many rational and irrational number in interval $(a;b)$. Again, let us prove this in two steps.

Lemma A. Between two arbitrary real numbers $x < y$ *there is at least one* $r \in \mathbf{Q}$, *i. e.* $x < r < y$.
Proof. Before conducting the proof itself, we must realize that this proposition is equivalent to the proposition that between two arbitrary real numbers there are infinitely many rational numbers. Assume that $x > 0$. Following Archimedean property, there certainly exists such natural number $n \in \mathbf{N}$ that $n > \frac{1}{y-x}$. After a few rearrangements we get $nx + 1 < ny$. Since $nx > 0$, there exists such $m \in \mathbf{N}$ that $m - 1 \leq nx < m$. Then $m \leq nx + 1 < ny$, and also $nx < m < ny$. This means that rational number $r = \frac{m}{n}$ is between *x* and *y*. If $x \leq 0$, take $k \in \mathbf{Z}$ such that $k > |x|$. Then numbers $x + k$, $y + k$ are positive, and similarly to the previous part, there exists $q \in \mathbf{Q}$ between them. Then rational number $r = q - k$ is between real numbers *x* and *y*.

Lemma B. Between two arbitrary real numbers $x < y$ *there is at least one* $w \in \mathbf{R} \setminus \mathbf{Q}$, *i. e.* $x < w < y$.
Proof. According to the previous lemma, between real numbers $\frac{x}{\sqrt{2}}$, $\frac{y}{\sqrt{2}}$ there is at least one rational number *r*. Consecutively, there must be irrational number $w = r\sqrt{2}$ between real numbers *x* and *y*, which is what we set out to prove.

We should now prove that if $r \neq 0$ is a rational number and $\alpha \in \mathbf{I}$, then $r\alpha \in \mathbf{I}$. This, however, is obvious, for if it were not true, then it would have to be $r\alpha \in \mathbf{Q}$, leading to $\alpha = \frac{r\alpha}{r} \in \mathbf{Q}$, which is a contradiction. In addition, we could again use the previous lemma and claim that there exists $q \in \mathbf{Q}$ such that $x - \sqrt{2} < q < y - \sqrt{2}$. Then the required $w = q + \sqrt{2}$.

Actually, we might now claim we have reached our objective. However, we seek to justify the importance of proofs in both mathematics and mathematics education. Therefore, we offer certain variability in proving the selected theorem. The previous multilayered proof required special notions, such as Archimedean property, primes, Euclid‘s proof etc. Let us present a more elementary proof of the theorem.

Another proof variant. Let $a \equiv p_0, p_1 p_2 \dots p_n \dots$; $b \equiv q_0, q_1 q_2 \dots q_n \dots$ (not considering decimal numbers with recurring 9). Approximations of these numbers are: p_0; p_0, p_1; $p_0, p_1 p_2$; ... and q_0; q_0, q_1; $q_0, q_1 q_2$; ... Let $a < b$, then in the first group of approximations there can be found such number $p_0, p_1 p_2 \dots p_k$ which is smaller than its corresponding number in the second group of approximations, that means $p_0, p_1 p_2 \dots p_{k-1} = q_0, q_1 q_2 \dots q_{k-1}$ and at the same time $p_k < q_k$. Consider number $\xi \equiv p_0, p_1 p_2 \dots p_k l_{k+1} l_{k+2} \dots$. Regardless of the values l_{k+1}, l_{k+2}, ... it is obvious that $\xi < b$. If $p_{k+1} \neq 9$, then let l_{k+1} be substituted by any arbitrary digit bigger than p_{k+1}; then regardless of the remaining digits it is certain that $a < \xi$. If $p_{k+1} = 9$ but $p_{k+2} < 9$, then let $l_{k+1} = 9$ and let digit l_{k+2} be bigger than p_{k+2} (or let this process be repeated). Since no all digits in the decimal string of number *a* are 9, it will always be possible to find such number $\xi > a$. Thus, $a < \xi < b$. Digits l_{k+1}, l_{k+2}, ... (starting from whichever) can be chosen arbitrarily, i. e. there are infinitely many numbers between *a* and *b*. If we choose any part of the decimal string of number ξ periodically, the result is a rational number; otherwise number ξ will be irrational.

CONCLUSIONS

Unfortunately, university first-year students' initial encounter with proofs in mathematics lectures is often extremely hard. They not only have to familiarize with new school environment, new education system, and new issues, but also with conceptual necessity to verify and prove all statements. Mathematical proofs make students' education even more difficult if we take them as a necessary evil, which is *x*-multiplied by the volume calculus textbooks. Perhaps the indispensability of proofs in mathematics study might be shown with the use of simple statements proofs, which could result in students' better understanding of proofs in more intricate mathematical theories.

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