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Original paper

Sum of generalized alternating harmonic series with three periodically repeated numerators

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ABSTRACT

This contribution deals with the generalized convergent harmonic series with three periodically repeated numerators; i. e. with periodically repeated numerators $(1, a, b)$, where $a, b \in \mathbb{R}$. Firstly, it is derived that the only value of the coefficient b, for which this series converges, is $b = -a - 1$. Then the formula for the sum $s(a)$ of this series is analytically derived. A relation for calculation the value of the constant $a \in \mathbb{R}$ from an arbitrary sum $s(a)$ also follows from the derived formula. The obtained analytical results are finally numerically verified by using the computer algebra system Maple 15 and its basic programming language.

KEYWORDS : harmonic series, alternating harmonic series, geometric series, sum of the series

JEL CLASSIFICATION: I30

INTRODUCTION

Let us recall the basic terms and notions. The *harmonic series* is the sum of reciprocals of all natural numbers except zero (see e.g. web page [4]), so this is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots .$$

The divergence of this series can be easily proved e.g. by using the integral test or the comparison test of convergence.

The series

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is known as the *alternating harmonic series*. This series converges by the alternating series test. In particular, the sum (interesting information about sum of series can be found e.g. in book [2] or paper [1]) is equal to the natural logarithm of 2:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2.$$

This formula is a special case of the *Mercator series*, the Taylor series for the natural logarithm:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

The series converges to the natural logarithm (shifted by 1) whenever $-1 < x \leq 1$.

MATERIAL AND METHODS

Sum of generalized alternating harmonic series with three periodically repeated numerators

We deal with the numerical series of the form

$$\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} + \frac{a}{3n-1} + \frac{b}{3n} \right) =$$

$$= \frac{1}{1} + \frac{a}{2} + \frac{b}{3} + \frac{1}{4} + \frac{a}{5} + \frac{b}{6} + \frac{1}{7} + \frac{a}{8} + \frac{b}{9} + \frac{1}{10} + \frac{a}{11} + \frac{b}{12} + \cdots , \qquad (1)$$

where $a, b \in \mathbb{R}$ are appropriate constants for which the series (1) converges. This series we shall call *generalized convergent harmonic series with periodically repeated numerators* $(1, a, b)$. We determine the values of the numerators a, b, for which the series (1) converges, and the sum of this series.

The power series corresponding to the series (1) has evidently the form

$$\sum_{n=1}^{\infty} \left(\frac{x^{3n-2}}{3n-2} + \frac{ax^{3n-1}}{3n-1} + \frac{bx^{3n}}{3n} \right) =$$

$$= \frac{x}{1} + \frac{ax^2}{2} + \frac{bx^3}{3} + \frac{x^4}{4} + \frac{ax^5}{5} + \frac{bx^6}{6} + \frac{x^7}{7} + \frac{ax^8}{8} + \frac{bx^9}{9} + \cdots . \qquad (2)$$

We denote its sum by $s(x)$. The series (2) is for $x \in (-1,1)$ absolutely convergent, so we can rearrange it and rewrite it in the form

$$s(x) = \sum_{n=1}^{\infty} \frac{x^{3n-2}}{3n-2} + a \sum_{n=1}^{\infty} \frac{x^{3n-1}}{3n-1} + b \sum_{n=1}^{\infty} \frac{x^{3n}}{3n} . \qquad (3)$$

If we differentiate the series (3) term-by-term, where $x \in (-1,1)$, we get

$$s'(x) = \sum_{n=1}^{\infty} x^{3n-3} + a \sum_{n=1}^{\infty} x^{3n-2} + b \sum_{n=1}^{\infty} x^{3n-1} . \qquad (4)$$

After reindexing and fine arrangement the series (4) for $x \in (-1,1)$ we obtain

$$s'(x) = \sum_{n=0}^{\infty} x^{3n} + ax\sum_{n=0}^{\infty} x^{3n} + bx^2\sum_{n=0}^{\infty} x^{3n},$$

that is

$$s'(x) = (1 + ax + bx^2)\sum_{n=0}^{\infty}(x^3)^n. \qquad (5)$$

When we summate the convergent geometric series (11) which has the first term 1 and the ratio x^3, where $|x^3| < 1$, i.e. for $x \in (-1,1)$, we get

$$s'(x) = (1 + ax + bx^2)\frac{1}{1 - x^3}.$$

We convert this fraction using the CAS Maple 15 to partial fractions and get

$$s'(x) = \frac{1+a+b}{3(1-x)} + \frac{(1+a-2b)x}{3(1+x+x^2)} + \frac{2-a-b}{3(1+x+x^2)},$$

where $x \in (-1,1)$. The sum $s(x)$ of the series (2) we obtain by integration in the form

$$s(x) = \int\left(\frac{1+a+b}{3(1-x)} + \frac{(1+a-2b)x}{3(1+x+x^2)} + \frac{2-a-b}{3(1+x+x^2)}\right)\mathrm{d}x =$$
$$= -\frac{1+a+b}{3}\ln(1-x) + \frac{1+a-2b}{6}\ln(1+x+x^2) +$$
$$+ \frac{1-a}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

From the condition $s(0) = 0$ we obtain

$$C = \frac{(a-1)\pi}{6\sqrt{3}},$$

hence

$$s(x) = -\frac{a+b+1}{3}\ln(1-x) + \frac{a-2b+1}{6}\ln(1+x+x^2)$$
$$-\frac{a-1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + \frac{(a-1)\pi}{6\sqrt{3}}. \qquad (6)$$

Now, we will deal with the convergence of the series (2) in the right point $x = 1$. After substitution $x = 1$ to the power series (2) – it can be done by the extended version of Abel's theorem (see [5], p. 23) – we get the numerical series (1). By the integral test we can prove that the series (1) converges if and only if $a + b + 1 = 0$. After simplification the equation (6), where $b = -a - 1$, we have

$$s(x) = \frac{1+a}{2}\ln(1+x+x^2) + \frac{1-a}{\sqrt{3}}\left(\arctan\frac{2x+1}{\sqrt{3}} - \frac{\pi}{6}\right).$$

For $x = 1$, because $\arctan(3/\sqrt{3}) = \pi/3$ and after re-mark $s(1)$ as $s(a)$, we get a simple formula

$$s(a) = \frac{1+a}{2}\ln 3 + \frac{\sqrt{3}(1-a)\pi}{18}. \qquad (7)$$

RESULTS AND DISCUSSION

Numerical results

We have solved the problem to determine the sum $s(a)$ above of the convergent numerical series (1) for several values of a (and for $b = -a - 1$) by using the basic programming language of the computer algebra system Maple 15. It was used the following simple procedure `sumgenhar1ab`:

```
sumgenhar1ab:=proc(t,a)
local r,k,s;
s:=0;
r:=0;
for k from 1 to t do
r:=1/(3*k-2) + a/(3*k-1) - (a+1)/(3*k);
s:=s+r;
end do;
print("t=",k-1,"s(",a,")=",evalf[20](s));
end proc:

A:=[0,1,2,3,4,5,6,7,8,9,10,100,-1,-2,-3,-4,-5,-6,-7,-8,       -9,-10,-
100,1/2,-1/2,
1/3,-1/3,1/4,-1/4,1/5,-1/5,1/6,-1/6,1/7,-1/7,1/8,-1/8,       1/8,1/9,-
1/9,1/10,-1/10]:
for a in A do
sumgenhar1ab(10000,a);
sumgenhar1ab(100000,a);
sumgenhar1ab(1000000,a);
end do;
```

The results of this procedure can be written into three following tables, where the number of the used triplets

$$\frac{1}{3n-2} + \frac{a}{3n-1} - \frac{a+1}{3n}$$

is denoted by t, the corresponding computed sum is denoted by $s(t, a)$ and the sums $s(a)$ are evaluated by means of the formula (7). The approximative values of the differences $|\Delta(a)|$ between the computed values of the sums $s(1000000, a)$ and the sums $s(a)$ and the relative quantification accuracy $r(a)$ of the sums $s(1000000, a)$, i.e. the ratio $|\Delta(a)/s(1000000, a)|$, are stated in two last columns of the tables 1, 2, and 3.

Very small difference both $|\Delta(-2)|$ and accuracy $r(-2)$ of the sum $s(1000000, -2)$ are caused by the fact that the triplet

$$\frac{1}{3n-2} - \frac{2}{3n-1} + \frac{1}{3n} = \frac{2}{(3n-2)(3n-1)3n}$$

is a fraction with small constant numerator independent of the variable n.

Table 1 The approximate values of the sums of the generalized harmonic series with three periodically repeating numerators $(1, a, -a-1)$ for some non-negative integers a

a	b	$s(10\,000, a)$	$s(100\,000, a)$	$s(1\,000\,000, a)$	$s(a)$	$\lvert\Delta(a)\rvert$	$r(a)$
0	−1	0.851 583 8	0.851 603 8	0.851 605 8	0.851 606 0	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
1	−2	1.098 579 0	1.098 609 0	1.098 612 0	1.098 612 3	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
2	−3	1.345 574 1	1.345 614 1	1.345 618 1	1.345 618 5	$4 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
3	−4	1.592 569 2	1.592 619 2	1.592 624 2	1.592 624 8	$6 \cdot 10^{-7}$	$4 \cdot 10^{-7}$
4	−5	1.839 564 4	1.839 624 4	1.839 630 4	1.839 631 0	$6 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
5	−6	2.086 559 5	2.086 629 5	2.086 636 5	2.086 637 3	$8 \cdot 10^{-7}$	$4 \cdot 10^{-7}$
6	−7	2.333 554 7	2.333 634 7	2.333 642 7	2.333 643 5	$8 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
7	−8	2.580 549 8	2.580 639 8	2.580 648 8	2.580 649 8	10^{-6}	$4 \cdot 10^{-7}$
8	−9	2.827 544 9	2.827 644 9	2.827 654 9	2.827 656 0	10^{-6}	$4 \cdot 10^{-7}$
9	−10	3.074 540 1	3.074 650 1	3.074 661 1	3.074 662 3	10^{-6}	$3 \cdot 10^{-7}$
10	−11	3.321 535 2	3.321 655 2	3.321 667 2	3.321 668 5	10^{-6}	$3 \cdot 10^{-7}$
100	−101	25.551 097 8	25.552 117 7	25.552 219 7	25.552 231 1	10^{-5}	$4 \cdot 10^{-7}$

Table 2 The approximate values of the sums of the generalized harmonic series with three periodically repeating numerators $(1, a, -a-1)$ for some negative integers a

a	b	$s(10\,000, a)$	$s(100\,000, a)$	$s(1\,000\,000, a)$	$s(a)$	$\lvert\Delta(a)\rvert$	$r(a)$
−1	0	0.604 588 7	0.604 598 7	0.604 599 7	0.604 599 8	10^{-7}	$2 \cdot 10^{-7}$
−2	1	0.357 593 5$\|$ $|$37 412 70	0.357 593 5$\|$ $|$37 779 35	0.357 593 5$\|$ $|$37 783 01	0.357 593 5$\|$ $|$37 783 05	$4 \cdot 10^{-14}$	10^{-13}
−3	2	0.110 598 4	0.110 588 4	0.110 587 4	0.110 587 3	10^{-7}	$2 \cdot 10^{-6}$
−4	3	−0.136 396 7	−0.136 416 7	−0.136 418 7	−0.136 419 0	$3 \cdot 10^{-7}$	$2 \cdot 10^{-6}$
−5	4	−0.383 391 9	−0.383 421 9	−0.383 424 9	−0.383 425 2	$3 \cdot 10^{-7}$	$8 \cdot 10^{-7}$
−6	5	−0.630 387 0	−0.630 427 0	−0.630 431 0	−0.630 431 5	$5 \cdot 10^{-7}$	$8 \cdot 10^{-7}$
−7	6	−0.877 382 2	−0.877 432 2	−0.877 437 2	−0.877 437 7	$5 \cdot 10^{-7}$	$6 \cdot 10^{-7}$
−8	7	−1.124 377 3	−1.124 437 3	−1.124 443 3	−1.124 444 0	$7 \cdot 10^{-7}$	$6 \cdot 10^{-7}$
−9	8	−1.371 372 4	−1.371 442 4	−1.371 449 4	−1.371 450 2	$8 \cdot 10^{-7}$	$6 \cdot 10^{-7}$
−10	9	−1.618 367 6	−1.618 447 6	−1.618 455 6	−1.618 456 5	$9 \cdot 10^{-7}$	$6 \cdot 10^{-7}$
−100	99	−23.847 930 1	−23.848 910 1	−23.849 008 1	−23.849 019 0	10^{-5}	$4 \cdot 10^{-7}$

Table 3 The approximate values of the sums of the generalized harmonic series with three periodically repeating numerators $(1, a, -a-1)$ for some a in the fractional form

a	b	$s(10\,000, a)$	$s(100\,000, a)$	$s(1\,000\,000, a)$	$s(a)$	$\lvert\Delta(a)\rvert$	$r(a)$
1/2	−3/2	0.975 081 4	0.975 106 4	0.975 108 9	0.975 109 2	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
1/3	−4/3	0.933 915 5	0.933 938 9	0.933 941 2	0.933 941 5	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
1/4	−5/4	0.913 332 6	0.913 355 1	0.913 357 4	0.913 357 6	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
1/5	−6/5	0.900 982 8	0.901 004 8	0.901 007 0	0.901 007 3	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
1/6	−7/6	0.892 749 7	0.892 771 3	0.892 773 5	0.892 773 7	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
1/7	−8/7	0.886 868 8	0.886 890 3	0.886 892 4	0.886 892 6	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
1/8	−9/8	0.882 458 2	0.882 479 5	0.882 481 6	0.882 481 8	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
1/9	−10/9	0.879 027 7	0.879 048 8	0.879 050 9	0.879 051 2	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
1/10	−11/10	0.876 283 3	0.876 304 3	0.876 306 4	0.876 306 7	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
−1/2	−1/2	0.728 086 2	0.728 101 2	0.728 102 7	0.728 102 9	$2 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
−1/3	−2/3	0.769 252 1	0.769 268 8	0.769 270 4	0.769 270 6	$2 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
−1/4	−3/4	0.789 835 0	0.789 852 5	0.789 854 3	0.789 854 5	$2 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
−1/5	−4/5	0.802 184 8	0.802 202 8	0.802 204 6	0.802 204 8	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
−1/6	−5/6	0.810 418 0	0.810 436 3	0.810 438 1	0.810 438 3	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
−1/7	−6/7	0.816 298 8	0.816 317 4	0.816 319 2	0.816 319 4	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
−1/8	−7/8	0.820 709 4	0.820 728 2	0.820 730 0	0.820 730 3	$3 \cdot 10^{-7}$	$4 \cdot 10^{-7}$
−1/9	−8/9	0.824 139 9	0.824 158 8	0.824 160 7	0.824 160 9	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
−1/10	−9/10	0.826 884 3	0.826 903 3	0.826 905 2	0.826 905 4	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$

Computation of 126 values $s(10000,a)$, $s(100000,a)$, $s(1000000,a)$ above took about 74 800 seconds, i.e. almost 20 hours 47 minutes. The relative quantification accuracies $r(a)$ of the sums $s(1000000,a)$ are, except the value $r(-2)$, approximately between $2\cdot 10^{-6}$ and $2\cdot 10^{-7}$.

CONCLUSIONS

In this paper we dealt with the generalized convergent harmonic series with three periodically repeated numerators $(1,a,b)$, where $a,b\in\mathbb{R}$, i.e. with the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{a}{3n-1}+\frac{b}{3n}\right)=\frac{1}{1}+\frac{a}{2}+\frac{b}{3}+\frac{1}{4}+\frac{a}{5}+\frac{b}{6}+\frac{1}{7}+\frac{a}{8}+\frac{b}{9}+\cdots.$$

We derived that the only value of the coefficient $b\in\mathbb{R}$, for which this series converges, is $b=-a-1$, and we also derived that the sum of this series is determined by the formula

$$s(a)=\frac{1+a}{2}\cdot\ln 3+\frac{\sqrt{3}(1-a)\pi}{18}.$$

This formula allows determine other sums whose three periodically repeated numerators need not be $(1,a,-a-1)$, but also $(k,l,-k-l)$ for arbitrary $k,l\in\mathbb{R}$, at least one nonzero. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{4}{3n-2}+\frac{1}{3n-1}-\frac{5}{3n}\right)=\frac{4}{1}+\frac{1}{2}-\frac{5}{3}+\frac{4}{4}+\frac{1}{5}-\frac{5}{6}+\frac{4}{7}+\frac{1}{8}-\frac{5}{9}+\cdots$$

has the sum

$$S(4,1,-5)=4\cdot S(1,1/4,-5/4)=4\cdot s(1/4)\doteq 6.3705.$$

Finally, we verified the main result by computing some sums by using the CAS Maple 15 and its basic programming language. These generalized alternating harmonic series so belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically and also presented by means of a simple numerical expression. From the derived formula above it follows that

$$a=\frac{6\sqrt{3}s(a)-3\sqrt{3}\ln 3-\pi}{3\sqrt{3}\ln 3-\pi}.$$

This relation allows calculate the value of the constant a for a given sum $s(a)$, as illustrates the following table:

Table 4 The approximate values of the constant a for some sums $s(a)$ of the generalized harmonic series with three periodically repeating numerators $(1,a,-a-1)$, where $\varphi=(1+\sqrt{5})/2=1.618033\ldots$

$s(a)$	a	$s(a)$	a	$s(a)$	a	$s(a)$	a	$s(a)$	a
1	0.600 770	6	20.843 173	−1	−7.496 191	−6	−27.738 594	0	1.098 612
2	4.649 251	7	24.891 653	−2	−11.544 672	−7	−31.787 074	π	9.270 966
3	8.697 731	8	28.940 134	−3	−15.593 152	−8	−35.835 555	e	7.557 201
4	12.746 212	9	32.988 614	−4	−19.641 633	−9	−39.884 035	φ	3.102 869
5	16.794 692	10	37.037 095	−5	−23.690 113	−10	−43.932 516	ln 3	1

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