{MERAA} Mathematics in Education, Research and Applications

Math Educ Res Appl, 2015(1), 2

Received: 2014-11-12
Accepted: 2015-02-06
Online published: 2015-11-16
DOI:

Original paper

Some notes on inverse matrix calculation

Norbert Kecskés[*]

Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Nitra, Slovak Republic

ABSTRACT
In the paper we point out and explain some misconceptions which frequently occur during hand calculation of the inverse matrix.

KEYWORDS: inverse matrix, elementary row and column operations, multiplication of matrices

JEL CLASSIFICATION: I21, J25

INTRODUCTION

Linear algebra plays a fundamental role both in theoretical and applied mathematics. The methods of this theory belong to the basic mathematical apparatus in the first years of studies at technical universities. When solving any linear problems, use of vectors, matrices, linear mappings etc. is absolutely necessary. This is the reason that thorough understanding and mastering of the methods of linear algebra is inevitable for future technical engineers.

MATERIAL AND METHODS

One of the easiest and most frequent methods of the inverse matrix calculation is the method based on elementary row operations (ERO). The principle of this methods can be indicated as

$$\left(A \mid E\right) \sim \ldots \overset{ERO}{} \ldots \sim \left(E \mid A^{-1}\right),$$

where A stands for the given matrix, A^{-1} for the inverse matrix and E for the unit matrix. So practically we proceed in a way that we augment the given matrix by the unit matrix, apply a sequence of ERO on both matrices simultaneously and convert the given matrix into the unit

[*] Corresponding author: Norbert Kecskés. Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
E-mail: norbert.kecskes@uniag.sk

matrix. This leads to the conversion of the unit matrix into the inverse matrix. Naturally, we suppose A be regular.

The simplest case is when $a_{11} = \pm 1$. In this case the calculation is straightforward and doesn't cause students problems (except for possible arithmetic mistakes). Some misunderstandings occur when $a_{11} \neq \pm 1$. For example, let's consider the matrix

$$A = \begin{pmatrix} -3 & 3 & 1 \\ 2 & 4 & 7 \\ 1 & 3 & 2 \end{pmatrix}$$

If we proceeded according to the abovementioned scheme we would have to multiply the first row by 2/3 or 1/3 and thus introduce fractions, which is not very convenient for hand calculation. For this reason it is good to have $a_{11} = \pm 1$ in our matrix. Note that by ERO we understand interchanging of rows, multiplying of rows by nonzero numbers and adding of arbitrary linear combination of rows to another row.
The question is: How does this affect the resulting inverse matrix since this matrix is unique.

RESULTS AND DISCUSSION

Let's consider the given matrix A and the task to find its inverse. We can have $a_{11} = \pm 1$ by two ways: by interchanging of the first and third row or by adding the second row to the first row. Probably the first way will be used more often. Let's denote this operation as R-1-3. The abovementioned misunderstandings arise with the questions that whether R-1-3 should be applied to the augmented matrix or just to the original matrix and how does it affect the inverse matrix and why it is so.
Hence (R-1-3 applied to the augmented matrix)

$$A = \left(\begin{array}{ccc|ccc} -3 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim^{R-1-3} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ -3 & 3 & 1 & 1 & 0 & 0 \end{array}\right) \tag{1}$$

or (R-1-3 applied only to the given matrix)

$$A = \left(\begin{array}{ccc|ccc} -3 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim^{R-1-3} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ -3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2}$$

In order to answer the question it is necessary to realize that behind every ERO or ECO (elementary column operation) stands a certain kind of matrix multiplication. It can be proved that every ERO is equivalent to multiplying of the given matrix from the left by a matrix which we got from the unit matrix by the same ERO. Analogously every ECO is equivalent to multiplying of the given matrix from the right by a matrix which we got from the unit matrix by the same ECO. Hence when performing ERO or ECO we perform some kind of virtual multiplying. For example our R-1-3 operation is equivalent to the product

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -3 & 3 & 1 \\ 2 & 4 & 7 \\ 1 & 3 & 2 \end{pmatrix}$$

This is the principle of the inverse matrix calculation using ERO. Let's consider (1), we have

$$(A \mid E),$$

and the R-1-3 operation can be symbolically written as

$$(C \cdot A \mid C \cdot E) = (A_1 \mid E_1).$$

We now convert the matrix A_1 into the unit matrix using ERO. Every ERO is equivalent to the abovementioned product hence we have a finite sequence of matrices $B_1, B_2 .... B_n$ for which

$$(B_n \cdot \ldots \cdot B_1 \cdot A_1 \mid B_n \cdot \ldots \cdot B_1 \cdot E_1).$$

It is obvious that $B_n \cdot \ldots \cdot B_1 = A_1^{-1}$ since on the left we want to have the unit matrix

$$(A_1^{-1} \cdot A_1 \mid A_1^{-1} \cdot E_1) = (E \mid A_1^{-1} \cdot E_1).$$

Further we have

$$(E \mid A_1^{-1} \cdot E_1) = (E \mid (C \cdot A)^{-1} \cdot C \cdot E) = (E \mid A^{-1} \cdot C^{-1} \cdot C \cdot E) = (E \mid A^{-1}).$$

We see that on the right we got the inverse matrix. So we can conclude that if we apply the required ERO (represented by the matrix *C*) on the augmented matrix the resulting matrix is directly the inverse matrix.
Now let's consider (2), so

$$(A \mid E)$$

and we perform the R-1-3 operation only on the matrix *A*

$$(C \cdot A \mid E) = (A_1 \mid E).$$

Now we again apply the sequence of ERO and convert the A_1 matrix into the unit matrix, in other words we multiply the A_1 matrix subsequently by the matrices $B_1, B_2 .... B_n$, so

$$(B_n \cdot \ldots \cdot B_1 \cdot A_1 \mid B_n \cdot \ldots \cdot B_1 \cdot E) = (A_1^{-1} \cdot A_1 \mid A_1^{-1} \cdot E) = (E \mid A_1^{-1} \cdot E) \text{ and}$$

$$(E \mid A_1^{-1} \cdot E) = (E \mid (C \cdot A)^{-1}) = (E \mid A^{-1} \cdot C^{-1}).$$

We see that on the right we didn't get the required inverse matrix but a C^{-1} multiple of it. In order to have A^{-1} we must multiply the matrix on the right by the matrix *C*, i.e.

$$\left(E \mid A^{-1} \cdot C^{-1} \cdot C\right) = \left(E \mid A^{-1}\right),$$

which is equivalent to ECO, in our case to interchanging of the first and third column. In other words $A^{-1} \cdot C^{-1}$ differs from A^{-1} in the first and third column. So if we carry out any ERO solely in the given matrix without altering the unit matrix, it is necessary to convert the resulting matrix on the right by respective ECO. Then we get the required inverse matrix.

Now let's consider the effect of ECO on the inverse matrix. Let's have again the matrix *A*. In order to have $a_{11} = \pm 1$, we can interchange the first and the third column and let's inspect the same problem as with the abovementioned ERO.
ECO applied on the augmented matrix

$$A = \left(\begin{array}{ccc|ccc} -3 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim^{C-1-3} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & -3 & 0 & 0 & 1 \\ 7 & 4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right) \qquad (3)$$

Note for (3) that an analogous ERO (1) led directly to the inverse matrix. And now ECO applied solely to the original matrix

$$A = \left(\begin{array}{ccc|ccc} -3 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim^{C-1-3} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & -3 & 1 & 0 & 0 \\ 7 & 4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \qquad (4)$$

For (3) we have

$$\left(A \mid E\right) \sim \left(A \cdot C \mid E \cdot C\right) \sim \left(A_1 \mid E_1\right) \sim \left(B_n \cdot \ldots \cdot B_1 \cdot A_1 \mid B_n \cdot \ldots \cdot B_1 \cdot E_1\right) \sim \left(A_1^{-1} \cdot A_1 \mid A_1^{-1} \cdot E_1\right) \sim$$
$$\sim \left(E \mid A_1^{-1} \cdot E_1\right) \sim \left(E \mid (A \cdot C)^{-1} \cdot E_1\right) \sim \left(E \mid C^{-1} \cdot A^{-1} \cdot E \cdot C\right) \sim \left(E \mid C^{-1} \cdot A^{-1} \cdot C\right)$$

In this case we see that we have to multiply the matrix on the right by the matrix *C* from the left and by the matrix C^{-1} from the right, i.e. we have to interchange the first and the third row and interchange the first and the third column.
And finally for (4)

$$\left(A \mid E\right) \sim \left(A \cdot C \mid E\right) \sim \left(A_1 \mid E\right) \sim \left(B_n \cdot \ldots \cdot B_1 \cdot A_1 \mid B_n \cdot \ldots \cdot B_1 \cdot E\right) \sim \left(A_1^{-1} \cdot A_1 \mid A_1^{-1} \cdot E\right) \sim$$
$$\sim \left(E \mid A_1^{-1}\right) \sim \left(E \mid (A \cdot C)^{-1}\right) \sim \left(E \mid C^{-1} \cdot A^{-1}\right)$$

In order to get the inverse matrix we have to multiply the matrix on the right by the matrix *C* from the left, i.e. to carry out the respective ERO, in our case to interchange the first and the third row.

CONCLUSIONS

In the paper we have illustrated how an ERO or ESO "pre-adjustment" of a matrix affects the resulting inverse matrix and discussed common mistakes related to this topic.

REFERENCES

[1] Bronson, R. (2010. Matrix operations. Schaum's Outline, McGraw Hill Professional. ISBN 0070079781.

[2] Demlová, N., Nagy, J. (1982). Algebra. SNTL Praha.

Reviewed by

1. Ing. Martina Hanová, PhD. Faculty of Economics and Management, Department of Statistics and Operations Research, Slovak University of Agriculture, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
2. Ing. Marcela Hallová, PhD. Faculty of Economics and Management, Department of Informatics, Slovak University of Agriculture, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic