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Original paper

"Problems" in determining the convergence of infinite series

Tomáš Pechočiak[1*], Lucia Rumanová[2]

[1] Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
[2] Constantine the Philosopher University in Nitra, Department of Mathematics, FNS, Tr. A. Hlinku 1, 949 74 Nitra, Slovak Republic

ABSTRACT
In this article we are presented different approaches to determine the convergence of infinite series. Students often do not know which criterion is appropriate for determining the convergence of the series. We don´t know always to determine the convergence of the series with using some criterions and so it is sometimes appropriate to use more of the given criterions.

KEYWORDS: infinite series, convergent series, convergence criteria

JEL CLASSIFICATION: I 35, D 45

INTRODUCTION

Within the subject Mathematics 2 the students of the Faculty of Engineering Slovak University of Agriculture in Nitra deal with the study of infinite numerical and functional series. The first we define why infinite series is convergent and then proceed with to determining the convergence of series. Students gain knowledge that serve to determine of convergence criteria. Students in solving of the task within the exercise or the test have often problem and they don´t know which criteria can be used. In some examples, that convergence series can be determined by various criteria. However, students often find that the criterion that they have chosen does not lead to the identification of convergence.

MATERIAL AND METHODS

Now we list the definition of infinite numerical series and the n-th partial sum.

Definition 1. When we wish to find the sum of an infinity sequence $\{a_n\}_{n=1}^{\infty}$, we call it an infinity series and write it in form

[*] Corresponding author: Tomáš Pechočiak. Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
E-mail: tomas.pechociak@uniag.sk

$$a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n \ .$$

The numbers a_n, $n = 1, 2, 3, ...$ we call n-th members of the series $\sum_{n=1}^{\infty} a_n$.

Definition 2. Number

$$s_n = a_1 + a_2 + a_3 + ... + a_n$$

we called n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$.

Now we define convergence series.

Definition 3. The sum of an infinite series is defined as the limit of the sequence of partial sums if the limit $\lim_{n\to\infty} s_n$ exists. The series is said to **converge** to a real number *s*, **diverge** or **diverge to** $\infty(-\infty)$**,** if the sequence of partial sums converges to *s*, $\lim_{n\to\infty} s_n = s$, or diverges to $\infty(-\infty)$, respectively, $\lim_{n\to\infty} s_n = \infty(-\infty)$.

The often is difficult when student with its knowledge does not know always determine n-th partial sum of the series and use it to check the convergence series. It is therefore necessary to determine the convergence another way. For its determination we use the convergence criteria. Now we list necessary condition of convergence series.

Theorem. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

This theorem is necessary condition of convergence series and the opposite theorem not valid, so if $\lim_{n\to\infty} a_n = 0$ then the series does not converges.

We will deal only series with non-negative members, so $a_i \geq 0$, for $i = 1,2,3,...$ and criteria for the determination of convergence. We most often use the following convergence criteria:

1. Comparison test
2. Cauchy limit criterion
3. D´Alembert limit criterion
4. Integral test

Now we give their definitions.

Comparison test:

Let the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with nonnegative terms and valid

$$a_1 \leq b_1,\ a_2 \leq b_2,\ \ldots,\ a_n \leq b_n,\ \ldots \ .$$

Then

a) if the infinite series $\sum_{n=1}^{\infty} b_n$ converges, then the infinite series $\sum_{n=1}^{\infty} a_n$ also converges,

b) if the infinite series $\sum_{n=1}^{\infty} b_n$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ also diverges.

When using this test, we comparative the series whit series $\sum_{n=1}^{\infty} q^n$ and $\sum_{n=1}^{\infty} \frac{1}{q^n}$, $q \neq 0$.

Cauchy limit criterion:

Let the series $\sum_{n=1}^{\infty} a_n$ is the series with nonnegative terms and there´s $\lim_{n\to\infty} \sqrt[n]{a_n} = q$. If

a) $q \in \langle 0,1)$, then the series converges,
b) $q > 1$, then the series diverges.

D´Alembert limit criterion:

Let the series $\sum_{n=1}^{\infty} a_n$ is the series with positive terms and there´s $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$. If

a) $q \in \langle 0,1)$, then the series converges,
b) $q > 1$, then the series diverges.

Integral test:

Let the series $\sum_{n=1}^{\infty} a_n$ is the series with nonnegative terms and let valid

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots \ .$$

Let $f(x)$ is continuous non-increasing function an interval $\langle 1, \infty)$, at which valid $f(n) = a_n$, for $n = 1, 2, 3, \ldots$. Let $\int_1^{\infty} f(x)dx = A$. If

a) A is number, then the series converges and for his sum s valid

$$a_1 \leq s \leq a_1 + A, \text{ or } A \leq s \leq a_1 + A,$$

b) A is ∞, then the series diverges.

RESULTS AND DISCUSSION

Now we give the task.

Task. Check the convergence of series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

Solution: First verify necessary condition for convergence of the series $\lim_{n\to\infty} a_n = 0$, therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+n} = 0,$$

thus series could be convergent. We have to find out with other criteria. As the n-th member is not n-th power, so student does not using Cauchy limit criterion. Probably the student not use comparison test because he does not know which the series should be compared. Students are “most like” D’Alembert limit criterion for its relative simplicity.

The n-th member of the series $a_n = \frac{1}{n^2+n}$ modifies to $a_n = \frac{1}{n(n+1)}$ and uses condition of D’Alembert criteria:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)\cdot(n+2)}}{\frac{1}{n\cdot(n+1)}}=\lim_{n\to\infty}\frac{n\cdot(n+1)}{(n+1)\cdot(n+2)}=\lim_{n\to\infty}\frac{n}{n+2}=1.$$

According to this criterion, we cannot decide whether it the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ is convergent or divergent, since $q=1$.

We must therefore determine convergence otherwise. The n-th member of the series modify as follows

$$a_n=\frac{1}{n^2+n}=\frac{1}{n(n+1)}=\frac{1+n-n}{n(n+1)}=\frac{1+n}{n(n+1)}-\frac{n}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

For the n-th partial sum of the series s_n is valid

$$s_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.$$

Then

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1.$$

It follows that a series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ is convergent and its sum $s=1$.

To determine the convergence of the series could have been used also integral test, because for the members of the series apply:

$$\frac{1}{2}>\frac{1}{6}>\frac{1}{12}>\frac{1}{20}>\ldots>\frac{1}{n^2+n}>\ldots.$$

The function $f:y=\frac{1}{x^2+x}$ is continuous and decreasing (hence non-increasing) on the interval $\langle 1,\infty)$ (see Figure 1).

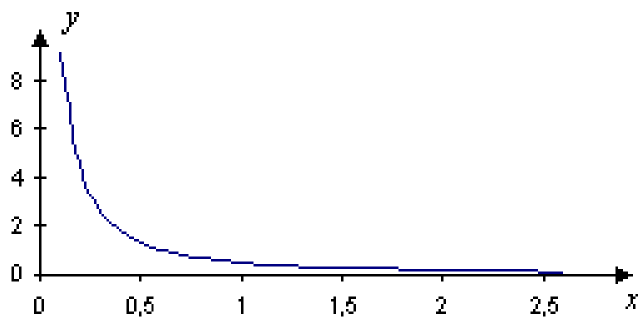

Fig. 1 Graph of the function $f:y=\frac{1}{x^2+x}$

Then

$$\int_1^{\infty}\frac{1}{x^2+x}dx=\int_1^{\infty}\frac{1}{x(x+1)}dx=\int_1^{\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=\lim_{b\to\infty}\left[\ln x-\ln(x+1)\right]_1^b=$$

$$= \lim_{b\to\infty}\left[\ln\frac{x}{x+1}\right]_1^b = \lim_{b\to\infty}\left[\ln\frac{b}{b+1} - \ln\frac{1}{2}\right] = \ln 1 - \ln 2^{-1} = \ln 2\,,$$

thus, a series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ of really converges for the sum valid:

$$\ln 2 \le s \le \frac{1}{2} + \ln 2\,,$$

or approximately $0{,}69 \le s \le 1{,}19$ ($s = 1$, see above).

CONCLUSIONS

In the article we showed different possibilities of determining the convergence of infinite numerical series. Students not only of the Faculty of Engineering Slovak University of Agriculture in Nitra have problems in dealing with such tasks. Students don't know choose the appropriate criterion for the determination, but they also have problems with the criteria themselves. The problem is that, these students come to the university from different types of secondary schools with different levels of mathematical knowledge and skills, many of them also have problems with mathematical operations and adjustments expressions, which occur in the various conditions of the given criteria.

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Reviewed by

1. Doc. Ing. Zuzana Poláková, PhD., Department of Statistics and Operations Research, Faculty of Economics and Management, Slovak University of Agriculture in Nitra, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
2. Doc. RNDr. Oleg Palumbíny, CSc., Institute of Applied Informatics, Automation and Mechatronics, Slovak university of technology in Bratislava, Hajdóczyho 1, 91724 Trnava, Slovak Republic